ACCURACY OF WENO AND ADAPTIVE ORDER WENO RECONSTRUCTIONS FOR SOLVING CONSERVATION LAWS∗

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Abstract. In this paper, we analyze standard WENO reconstructions and multilevel WENO reconstructions with adaptive order (WENO-AO) using both WENO-JS and WENO-Z weighting. We also present a new WENO-AO reconstruction. We give conditions under which the reconstructions achieve optimal order accuracy for both smooth solutions and solutions with shocks. The old WENO-AO reconstruction drops to a fixed, base level of approximation when there are shocks in the solution, but the new one maintains the accuracy of the largest stencil over which the solution is smooth. Our analysis in the discontinuous case requires that the smoothness indicators do not approach zero as the grid is refined. We provide a condition to ensure this result, but we also show an example where this can fail to occur. That is, we show that WENO reconstructions can fail to maintain the order of approximation of the smallest stencil over which the solution is smooth. We also present numerical results confirming the convergence theory of the old and new WENO-AO reconstructions, and compare their performance in solving conservation laws.

Key words. Weighted essentially non-oscillatory, WENO-AO, polynomial reconstruction, hyperbolic equation, approximation, shock discontinuity

AMS subject classifications. 65D05, 65M08, 65M06, 65M12, 76M12, 76M20

1. Introduction. For solving of a system of hyperbolic conservation laws,

\begin{equation}
    u_t + f(u)_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^d, \quad d \geq 1,
\end{equation}

weighted essentially non-oscillatory (WENO) schemes [5, 11, 10] are a popular choice. They allow one to reconstruct a high order version of the solution merely from approximations of cell averages (in finite volume schemes) or point values (in finite difference schemes). The key is to average approximations defined on various stencils, and to weight them so as to avoid stencils containing a shock discontinuity in the solution. The idea is that high order accuracy may be achieved by the approximation on the big stencil where the solution is smooth, and yet reduce to the order of the smaller stencils when there is a shock to avoid.

It appears that standard WENO reconstructions were not proved to have this property until 2011, when Arándiga, Baeza, Belda, and Mulet gave a proof [1]. They clarified the delicate nature of the parameters \( \epsilon \) and \( \eta \) in the standard nonlinear weighting procedure, WENO-JS, of Jiang and Shu [7]. In particular, \( \epsilon \) needs to be proportional to \( h^2 \), where \( h \) is the the grid spacing, and \( \eta > s/2 \), where the low order polynomials approximate to order \( s \). Kolb [8] later analyzed the CWENO3 scheme of Levy, Puppo, and Russo [9]. This latter paper provided a new way to obtain WENO reconstructions involving combining polynomials of different degrees. It was

∗This work was supported in part by the U.S. National Science Foundation grant DMS-1418752, the Taiwan Ministry of Science and Technology grant MOST 105-2115-M-110-006-MY2 through the Multidisciplinary and Data Science Research Center of the National Sun Yat-sen University, and the National Center for Theoretical Sciences, Taiwan.

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In this paper, we analyze the accuracy of the standard WENO and WENO-AO reconstructions. We include results for when the WENO-Z weighting procedure of Castro, Costa, and Don [4, 3] is used. Our results are summarized in Sections 4.3.5 and 5.3 below. The standard WENO reconstruction behaves as desired, even with WENO-Z weights. It is high order accurate when the solution is smooth, and it drops to low order \( s \) when there is a discontinuity (not on the central cell), provided only that \( \eta \geq s/2 \). We show that this condition is sharp. The two-level WENO-AO\((r,s)\) reconstructions behave similarly. They can achieve higher order \( r \) accuracy in the smooth case and otherwise maintain at least order \( s \) accuracy, provided that when using WENO-JS weights, \( r \leq 2s - 1 \) and \( \eta \geq s/2 \). WENO-Z weights are more complex, because the accuracy of the reconstruction depends more strongly on the choice of \( \eta \), as we will show.

Multilevel WENO-AO\(_s(r_\ell, \ldots, r_1, s)\) reconstructions [2] are based on a base state with approximation order \( s \). When the solution is smooth the approximation attains the highest order \( r_\ell \), but when it is discontinuous, it usually drops to the base level \( s \). When using WENO-JS weights, one requires \( r_\ell \leq 2s - 1 \) and \( \eta \geq s/2 \), which is equivalent to the two-level case. That is, from the point of view of approximation order, there is no point in using the multilevel reconstruction. Again, WENO-Z weights are more complex. There is a restriction on the size of the gap between successive approximation levels, but a careful choice of \( \eta \) can allow any order. However, the accuracy almost always drops to order \( s \) when the solution is discontinuous.

We present a new multilevel WENO-AO\((r_\ell, \ldots, r_1, r_0)\) reconstruction that has no base level. When the solution is smooth the approximation attains the highest order \( r_\ell \), but when it is discontinuous, it drops to the order of the largest stencil that does not contain the discontinuity.

When the solution has a discontinuity, our convergence results require the smoothness indicators \( \sigma \to 0 \) as the grid is refined, as is the case in [1, 8]. We show that this hypothesis can fail, and replace it by the hypothesis that the discontinuity is bounded away from the gridpoints as the grid is refined (Definition 5.1). We further show that there are sequences of grids for which even this hypothesis fails. In that case, it is possible that a WENO reconstruction fails to maintain the lowest order of approximation. This seems contrary to the prevailing understanding of WENO reconstructions appearing in the literature.

The paper is structured as follows. In the next section, we give the background needed to understand WENO reconstruction techniques. For the knowledgable reader, this section, and the next, set our notation. In Section 3, we define the various WENO reconstructions and define our new one. Section 4 presents a rigorous analysis of the errors in the reconstructions when the solution is smooth. We provide conditions needed to ensure that the reconstruction achieves the order of accuracy of the big stencil. In Section 5, we give our analysis for the case when the solution has a discontinuity. Our cautionary example of a situation where a WENO reconstruction fails to maintain the lowest order of approximation appears in Section 6. Finally, numerical results comparing the old and new WENO-AO reconstructions are given in the last section.

2. Background on WENO reconstructions. We first review the background setting required for WENO reconstructions. For simplicity of exposition, we consider the finite volume framework. The finite difference reconstruction is similar.
Partition space by grid points $\cdots < x_{-1} < x_0 < x_1 < \cdots$. Define the cell $I_i = [x_i, x_{i+1}]$, its length $\Delta x_i = x_{i+1} - x_i$, and its midpoint $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$. Let $h = \max_i \Delta x_i$ and assume that the grid is quasiuniform (i.e., there is some $\rho > 0$ such that $\rho h \leq \min_i \Delta x_i$). Let $\bar{u}_i$ be the average of $u(x)$ on the cell $I_i$, i.e.,

$$\bar{u}_i = \frac{1}{\Delta x_i} \int_{I_i} u(x) \, dx. \tag{2.1}$$

### 2.1. Polynomial approximation on stencils.

Now assume that $u$ is smooth. For an $r$th order approximation of $u$ on a given cell $I_i$, we consider the ordered stencil $S^r_j(i)$, which contains $r$ cells, including $I_i$, and is defined by

$$S^r_j(i) = \{I_{i+j-\lfloor\frac{r}{2}\rfloor}, \ldots, I_i, \ldots, I_{i+j+\lfloor\frac{r}{2}\rfloor}\}, \tag{2.2}$$

where $-\lfloor\frac{r-1}{2}\rfloor \leq j \leq \lfloor\frac{r}{2}\rfloor$. Moreover, $S^r_0$ denotes the central stencil. From now and for the remainder of the paper, we fix a value of $i$ and drop it from the notation. For each $S^r_j$, we obtain the $r$th order stencil polynomial $P^r_j$ (a polynomial of degree $r - 1$) by imposing the interpolation conditions

$$\frac{1}{\Delta x_k} \int_{I_k} P^r_j(x) \, dx = \bar{u}_k \quad \forall I_k \in S^r_j. \tag{2.3}$$

### 2.2. Smoothness indicators.

The smoothness indicator $\sigma$ defined by Jiang and Shu in [7] is normally used to measure the smoothness of stencil polynomials on the cell $I_i$. For the stencil $S^r_j$, it is given by

$$\sigma^r_j = \sum_{\ell=1}^{r-1} \int_{I_i} (\Delta x_i)^{2\ell-1} \left(\frac{d^{\ell} P^r_j(x)}{dx^{\ell}}\right)^2 \, dx. \tag{2.4}$$

If the grid is uniform in the cell size $\Delta x = h$, then in regions where $u$ is smooth, the Taylor expansion of (2.4) gives

$$\sigma^r_j = (u'h)^2 + O(h^4), \tag{2.5}$$

which is $O(h^2)$ or $O(h^4)$ at a critical point. If there are discontinuities in $u$ within the stencil $S^r_j$, then $\sigma^r_j = O(1)$. Summarizing,

$$\sigma^r_j = \begin{cases} O(h^2), & \text{if } u \text{ is smooth on } S^r_j, \\ O(1), & \text{if } u \text{ has a jump discontinuity on } S^r_j. \end{cases} \tag{2.6}$$

### 2.3. Nonlinear weights.

A WENO reconstruction is a weighted sum of stencil polynomials. For the stencil $S^r_j$, let us denote its (constant) weight as $\alpha^r_j$. We refer to this weight as a linear weight. For the linear weight $\alpha^r_j$, we define its nonlinear weight $\tilde{\alpha}^r_j$, as discussed below.

For some collection of stencils, we merely require that the linear weights sum to one, i.e., $\sum_{r,j} \alpha^r_j = 1$. (We also desire that $\alpha^r_j > 0$.) When the solution $u$ is smooth, the linear weights should be chosen so as to give a high order WENO reconstruction. However, when there are discontinuities in the data over some stencils, we want to bias the weighted sum so as to exclude those stencils. The idea is to make $\tilde{\alpha}^r_j \simeq \alpha^r_j$ when $u$ is smooth on $S^r_j$ and to make $\tilde{\alpha}^r_j \simeq 0$ when $u$ is not smooth on $S^r_j$. 


2.3.1. WENO-JS weights. For complete generality, let the linear weights be \( \alpha_k^s \) for various stencils indexed by \( k \) of size \( s \). To define the nonlinear weights, Jiang and Shu [7] scale each linear weight \( \alpha_j^r \) as

\[
\hat{\alpha}_j^r = \frac{\alpha_j^r}{(\sigma_j^r + \epsilon_h)\eta'},
\]

and then normalize

\[
\tilde{\alpha}_j^r = \frac{\hat{\alpha}_j^r}{\sum_{k,s} \hat{\alpha}_k^s},
\]

for some exponent \( \eta \geq 1 \) and \( \epsilon_h > 0 \), which avoids any possibility of division by zero. Usually, one takes \( \eta = 2 \) and \( \epsilon_h = \epsilon \approx 10^{-6} \). Arándiga et al. [1] show that, in fact, \( \eta \) must be chosen carefully and that it is valuable to take \( \epsilon_h = Kh^2 \), for some fixed \( K > 0 \). We will see this in Sections 4–5.

2.3.2. WENO-Z weights. For standard WENO, Castro et al. give the general formula for the WENO-Z reconstruction in [4], for \( r \geq 3 \). First let \( m = \lfloor \frac{r}{2} \rfloor \) and use the smoothness indicators to define

\[
\tau = \begin{cases} 
|\sigma_{r-m} - \sigma_m|, & \text{if } r \text{ is odd}, \\
|\sigma_{r-m+1} - \sigma_{r-m+2} - \sigma_{r-m-1} + \sigma_m|, & \text{if } r \text{ is even}.
\end{cases}
\]

Then the un-normalized nonlinear weights are, for \( \eta \geq 1 \),

\[
\hat{\alpha}_j^r = \alpha_j^r \left( 1 + \left( \frac{\tau}{\sigma_j^r + \epsilon_h} \right)^\eta \right),
\]

and the normalized weights are given by (2.8).

3. WENO reconstructions. In this section, we review the standard and adaptive order WENO reconstructions. We also present our new adaptive order WENO reconstruction.

3.1. Standard WENO reconstruction. Suppose we are interested in an \( r = (2s - 1) \)st order standard WENO reconstruction for \( s \geq 2 \). First consider all the small stencils with \( s \) cells containing the given cell \( I_i \). For each \( S_j^s \), we construct \( P_j^s \). Moreover, we define the big stencil \( S_0^r = \bigcup_j S_j^s \) and construct a higher order polynomial \( P_0^r \) on it. At a fixed point \( x^* \), the polynomial \( P_0^r \) can often be written as a convex combination of \( P_j^s \), so

\[
P_0^r(x^*) = \sum_j \alpha_j^s P_j^s(x^*),
\]

where \( \sum_j \alpha_j^s = 1 \). We refer to \( \alpha_j^s \) as an exact linear weight. These weights can be precomputed for the given \( x^* \) (if they do indeed exist).

When there are discontinuities in the data over the big stencil \( S_0^r \), we want to make use of the relatively small stencils on which \( u \) is smooth in order to achieve the essentially non-oscillatory property. The standard WENO reconstruction, valid only for \( x = x^* \), is

\[
R_r(x^*) = \sum_j \tilde{\alpha}_j^s P_j^s(x^*).
\]
3.2. WENO reconstructions with adaptive order (WENO-AO). In [9], Levy, Pupo and Russo describe a third order compact, Central WENO scheme (CWENO). They use a somewhat different WENO reconstruction than the standard one (3.2), because the linear weights in (3.1) fail to exist when \( r = 2 \) (\( 2r - 1 = 3 \)) and \( x^s = x_{i+\frac{3}{2}} \). For the given cell \( I_i \), they combine the optimal quadratic polynomial \( P^5_0 \) and two linear polynomials \( P^3_2 \) and \( P^3_0 \). Three advantages of their approach are that exact linear weights are not required, the weights can be taken to be positive, and the reconstruction holds for any point \( x \in I_i \). The disadvantage is that the big stencil polynomial must be computed.

Zhu and Qiu describe a fifth order WENO reconstruction in [13], where they combined the fifth order stencil polynomial with two linear stencil polynomials. Later, Balsara, Garain and Shu introduced a new class of WENO reconstructions with adaptive order in [2], which we will briefly recall below. The idea is to combine the three quadratic stencil polynomials with some higher order stencil polynomials.

3.2.1. Two-level WENO-AO reconstruction. The fifth order reconstruction WENO-AO(5,3) is based on the large stencil \( S^5_0 = \{ I_{i-2}, I_{i-1}, I_{i}, I_{i+1}, I_{i+2} \} \) and the three small stencils \( S^3_{-1} = \{ I_{i-2}, I_{i-1}, I_{i} \}, S^3_0 = \{ I_{i-1}, I_{i}, I_{i+1} \}, S^3_1 = \{ I_{i}, I_{i+1}, I_{i+2} \}, \) from which we obtain the stencil polynomials \( P^5_0, P^3_{-1}, P^3_0 \), and, \( P^3_1 \), respectively. The reconstruction is given by

\[
R^{5,3}(x) = \frac{\alpha^5_0}{\alpha^5_0} [P^5_0(x) - \sum_{j=-1}^{1} \alpha^3_j P^3_j(x)] + \sum_{j=-1}^{1} \alpha^3_j P^3_j(x),
\]

where \( \alpha^5_0 \) and \( \alpha^3_j \), \( j = -1, 0, 1, \) are arbitrary positive linear weights such that \( \alpha^5_0 + \sum_j \alpha^3_j = 1 \).

Using the same idea, WENO-AO(7,3) is defined on the big stencil \( S^7_0 \) and the same small stencils \( S^3_{-1}, S^3_0, S^3_1 \), and WENO-AO(9,3) is given on \( S^9_0, S^9_{-1}, S^9_0, S^9_1 \). In analogy with (3.3), the general formulation of WENO-AO(r, s) reconstruction, \( r > s \geq 2 \), can be written as

\[
R^{r,s}(x) = R^{r,s}_0(x) = \frac{\alpha^r_0}{\alpha^r_0} [P^r_0(x) - \sum_{j} \alpha^s_j P^s_j(x)] + \sum_{j} \alpha^s_j P^s_j(x),
\]

where \( S^r_0 \supseteq \bigcup_j S^s_j \neq \emptyset \). The positive linear weights \( \alpha^r_0 \) and \( \alpha^s_j \) can be chosen arbitrarily up to requiring \( \alpha^r_0 + \sum \alpha^s_j = 1 \). If WENO-Z weights are used, following [2], the definition of \( \tau \) is generalized to

\[
\tau = \frac{1}{\# \text{ of } j} \sum_j |\sigma^r_0 - \sigma^s_j|.
\]

3.2.2. Multilevel WENO-AO reconstruction. It is possible that the solution on \( S^7_0 \) is non-smooth but \( S^5_0 \) gives a smooth solution, so Balsara et al. [2] combine \( R^{7,3} \) and \( R^{5,3} \). The algorithm is given by

\[
R^{7,5,3}_3(x) = \frac{\gamma^{7,3}}{\gamma^{7,3}} [R^{7,3}_3(x) - \gamma^{5,3} R^{5,3}_3(x)] + \gamma^{5,3} R^{5,3}_3(x),
\]

where \( \gamma^{7,3} + \gamma^{5,3} = 1 \) and \( \gamma^{7,3} > 0, \gamma^{5,3} > 0 \) are arbitrary.

Similarly, WENO-AO(9,7,5,3) is defined by \( R^{9,3}_4 \) and \( R^{5,3}_4 \). Using this recursive process WENO-AO(9,7,5,3) combines \( R^{9,3}_4 \) and \( R^{5,3}_4 \) [2], where each reconstruction...
includes the base order 3 polynomials. The generalized recursion formula of multilevel WENO-AO($r_\ell, r_{\ell-1}, \ldots, r_1, s$), $\ell \geq 1$, for approximation levels $r_\ell > r_{\ell-1} > \cdots > r_1 > s \geq 2$ and base level $s$, is (3.4) for $R_{s}^{r_s-s}(x)$ for all $1 \leq k \leq \ell$ and

$$
R_{s}^{r_\ell, r_{\ell-1}, \ldots, r_1, s}(x) = \frac{\gamma_{r_\ell-s}}{\gamma_{r_\ell-s}} \left[ R_{s}^{r_s}(x) - \gamma_{r_{\ell-1}} \cdots, r_1, s \right] R_{s}^{r_{\ell-1}} \cdots, r_1, s(x) \]
$$

(3.7)

where $S_0^r \supset S_0^{r_{\ell-1}} \supset \cdots \supset S_0^s \supset \cup_j S_0^j \neq \emptyset$. We could further generalize (3.7) to include all pertinent stencils, i.e., all $S_{r}^{m} \subset S_0^r$, but we omit the details.

For WENO-Z weights, when $\gamma_{r_\ell-s} > 0$ and $\gamma_{r_{\ell-1}} \cdots, r_1, s > 0$ are arbitrary such that $\gamma_{r_\ell-s} + \gamma_{r_{\ell-1}} \cdots, r_1, s = 1$. We define the WENO-JS nonlinear weights through the un-normalized weighting

$$
\hat{\gamma}_{r_\ell-s} = \frac{\gamma_{r_\ell-s}}{\sigma_0^{r_\ell-s} + \epsilon_h \sigma_0^{r_{\ell-1}}}, \quad \hat{\gamma}_{r_{\ell-1}} \cdots, r_1, s = \frac{\gamma_{r_{\ell-1}} \cdots, r_1, s}{(\sigma_0^{r_{\ell-1}} + \epsilon_h \sigma_0^{r_{\ell-1}})},
$$

(3.8)

and the normalized nonlinear weights are then

$$
\hat{\gamma}_{r_\ell-s} = \frac{\hat{\gamma}_{r_\ell-s}}{\hat{\gamma}_{r_{\ell-1}} \cdots, r_1, s}, \quad \hat{\gamma}_{r_{\ell-1}} \cdots, r_1, s = \frac{\hat{\gamma}_{r_{\ell-1}} \cdots, r_1, s}{\hat{\gamma}_{r_\ell-s} + \hat{\gamma}_{r_{\ell-1}} \cdots, r_1, s}.
$$

(3.9)

For WENO-Z weights, when $\ell \geq 2$, define

$$
\tau = |\sigma_0^{r_\ell} - \sigma_0^{r_{\ell-1}}|,
$$

(3.10)

the un-normalized weights

$$
\hat{\gamma}_{r_\ell-s} = \gamma_{r_\ell-s} \left( 1 + \left( \frac{\tau}{\sigma_0^{r_\ell} + \epsilon_h \sigma_0^{r_{\ell-1}}} \right)^v \right),
$$

(3.11)

and the normalized nonlinear weights by (3.9). Note that $\sigma_0^{r_\ell}$ and $\sigma_0^{r_{\ell-1}}$ from the larger stencils are used in (3.8) and (3.11).

We can certainly use different values of $\eta$ at each stage of the reconstruction. We will find this useful for the WENO-Z weights. In this case, we use $\eta_{0,k}$ in the initial stage (3.4) for $R_{s}^{r_s-s}(x)$, and we use $\eta_\ell$ in (3.11).

3.2.3. A new multilevel WENO-AO reconstruction. The original multilevel WENO-AO constructions $R_{s}^{r_\ell, r_{\ell-1}, \ldots, r_1, s}$ in (3.7) are based on $R_{s}^{r_s-s}$, $1 \leq k \leq \ell$; that is, at each level, the reconstructions may revert back to the base level $s$. As we will see, for each $k$, when $u$ is smooth on $S_0^s$, $r_k$ and $s$ need to satisfy Theorem 5.7 below to have order $r_k$ accuracy. Moreover, when $u$ has a discontinuity on the two biggest stencils, Theorem 5.7 below shows that the order of accuracy is at best the base level $s$. Our goal is to define a new reconstruction that has no base level, and thereby has relaxed constraints on the levels needed for accuracy and achieves a higher order of accuracy near discontinuities.

Define $R_{s}^{r_\ell, r_{\ell-1}, \ldots, r_0}$ as in (3.4), which uses the stencils $S_0^s$ and $S_j^s$ for several $j$. The new multilevel WENO-AO reconstruction has no base level, and it is denoted $R_{s}^{r_\ell, r_{\ell-1}, \ldots, r_0}$,
where \( r_\ell > r_{\ell-1} > \cdots > r_0 \geq 2 \). It is given recursively for \( \ell \geq 2 \) by

\[
R_{r_r,r_{r-1},\ldots,r_0}(x) = \frac{\tilde{\alpha}_0^{r_\ell}}{\alpha_0^{r_\ell}} \left[ P_0^{r_\ell}(x) - \left( \sum_j \alpha_j^{r_0} \right) R_{r_{r-1},\ldots,r_0}(x) \right] \\
+ \left( \sum_j \tilde{\alpha}_j^{r_0} \right) R_{r_{r-1},\ldots,r_0}(x),
\]

(3.12)

where \( S_0^{r_\ell} \supset S_0^{r_{r-1}} \supset \cdots \supset S_0^{r_0} \supset \cup_j S_j^r \neq \emptyset \). Again, we could generalize this to include all pertinent stencils, but we do not pursue this here. The linear weights \( \alpha_0^{r_\ell} > 0 \) and \( \alpha_j^{r_0} > 0 \), for all \( j \), are arbitrary such that \( \alpha_0^{r_\ell} + \sum_j \alpha_j^{r_0} = 1 \). For WENO-Z weights, we define \( \tau \) by (3.10). Compared to (3.7), note that here we use \( P_0^{r_\ell} \) instead of \( R^{r_r-r_0} \) and we use the smoothness indicators \( \sigma^{r_\ell} \) and \( \sigma_j^{r_0} \) for the nonlinear weighting. For example, the new multilevel WENO-AO(7,5,3) is defined as

\[
R^{7,5,3}(x) = \frac{\tilde{\alpha}_0^7}{\alpha_0^7} P_0^7(x) - \left( \sum_{j=-1}^1 \alpha_j^3 \right) R^{5,3}(x) + \left( \sum_{j=-1}^1 \tilde{\alpha}_j^3 \right) R^{5,3}(x),
\]

(3.13)

where \( \tilde{\alpha}_0^7 + \sum_j \tilde{\alpha}_j^3 = 1 \) and \( R^{5,3}(x) \) is given by (3.3).

4. Accuracy analysis when \( u \) is smooth. In this section, we give a rigorous analysis of accuracy of WENO reconstructions in the case where \( u \) is smooth on the large stencil. We show under what conditions they give the desired accuracy. Our results can be viewed as generalizations of those in [1], where the authors analyzed standard WENO reconstructions, and in [8], where the author proved the accuracy for the compact CWENO3 scheme and its reconstruction. Both papers considered only WENO-JS weights.

At times, we require a tight assessment of asymptotic behavior. Recall that for a function \( f(h) \),

\[
f = \mathcal{O}(h^r) \iff |h^{-r} f| \leq C \text{ as } h \to 0,
\]

(4.1)

for some constant \( C > 0 \). The notation \( f = \Theta(h) \) provides upper and lower bounds:

\[
f = \Theta(h^r) \iff C_1 h^r \leq |f| \leq C_2 h^r \text{ as } h \to 0,
\]

(4.2)

for some positive constants \( C_1 \) and \( C_2 \).

4.1. Smoothness indicators. We begin by looking at the smoothness indicators. The following lemma appears in [1] when comparing smoothness indicators on the same size stencils. When the stencil sizes differ, we have the result of Kolb [8], which deals only with the compact CWENO3 reconstruction. We provide a simple proof that covers all cases.

**Lemma 4.1.** Let cell \( I_i \) and any stencils \( S^r \ni I_i \) and \( S^s \ni I_i \) be given (actually \( S^r_j \) and \( S^s_k \), but the offsets \( j \) and \( k \) are immaterial). For \( r \geq s \geq 2 \), assume \( P^r \) and \( P^s \) are stencil polynomials from \( S^r \) and \( S^s \), respectively. If the smoothness indicators \( \sigma^s \) and \( \sigma^r \) are given by (2.4), then

\[
\sigma^r - \sigma^s = \mathcal{O}(h^{r+1}),
\]

(4.3)

provided that \( u \) is smooth on \( S^r \cup S^s \).
Proof. First we have that, for any $\ell = 0, 1, \ldots, s$,
\begin{equation}
\frac{d^\ell}{dx^\ell} (P^r - P^s) = \frac{d^\ell}{dx^\ell} (P^r - u) - \frac{d^\ell}{dx^\ell} (P^s - u) = O(h^{s-\ell}).
\end{equation}
Since
\begin{equation}
\left( \frac{d^\ell}{dx^\ell} P^r \right)^2 = \left( \frac{d^\ell}{dx^\ell} P^s + \frac{d^\ell}{dx^\ell} (P^r - P^s) \right)^2
= \left( \frac{d^\ell}{dx^\ell} P^s \right)^2 + \left( \frac{d^\ell}{dx^\ell} (P^r - P^s) \right)^2 + 2 \frac{d^\ell}{dx^\ell} P^s \frac{d^\ell}{dx^\ell} (P^r - P^s),
\end{equation}
we have that
\begin{equation}
\sigma^r - \sigma^s = \sum_{\ell=1}^r \Delta x_i^{2\ell-1} \int_{I_\ell} \left[ \left( \frac{d^\ell}{dx^\ell} P^r \right)^2 - \left( \frac{d^\ell}{dx^\ell} P^s \right)^2 \right] dx
= \sum_{\ell=1}^r \Delta x_i^{2\ell-1} \int_{I_\ell} \left[ \left( \frac{d^\ell}{dx^\ell} (P^r - P^s) \right)^2 + 2 \frac{d^\ell}{dx^\ell} P^s \frac{d^\ell}{dx^\ell} (P^r - P^s) \right] dx
= \sum_{\ell=1}^r \Delta x_i^{2\ell-1} \int_{I_\ell} \left[ O(h^{s-\ell})^2 + 2 O(1) O(h^{s-\ell}) \right] dx
= O(h^{s+1}).
\end{equation}

4.2. Nonlinear weights. The following theorem quantifies the perturbation of the nonlinear weights from the linear ones.

**Theorem 4.2.** Let $\eta \geq 1$, $\epsilon > 0$, and $K > 0$. Let cell $I_i$ and $\ell \geq 1$ be given. For a collection of $\ell + 1$ stencils $S^{\tau_k} \ni I_i$, $k = 0, 1, \ldots, \ell$, where $2 \leq \tau_0 \leq \tau_1 \leq \ldots \leq \tau_\ell$, let $\alpha^{\tau_k}$ be positive linear weights such that $\sum_k \alpha^{\tau_k} = 1$, and let $\sigma^{\tau_k}$ be the corresponding smoothness indicators. If $u$ is smooth on $\bigcup_{k=0}^\ell S^{\tau_k}$, then the following hold.

1. **WENO-JS weights** satisfy, for all $k = 0, 1, \ldots, \ell$,
\begin{equation}
\tilde{\alpha}^{\tau_k} = \alpha^{\tau_k} + \begin{cases} O(h^{\tau_k+1}), & \text{if } \epsilon_h = \epsilon, \\
O(h^{\tau_k+1}), & \text{if } \epsilon_h = K h^2. \end{cases}
\end{equation}

2. **WENO-Z weights** satisfy, for all $k = 0, 1, \ldots, \ell$,
\begin{equation}
\tilde{\alpha}^{\tau_k} = \alpha^{\tau_k} + \begin{cases} O(h^{\tau_k+1+(\tau_m+1)\eta}), & \text{if } \epsilon_h = \epsilon, \\
O(h^{\tau_k+1+(\tau_m+1)\eta}), & \text{if } \epsilon_h = K h^2, \end{cases}
\end{equation}

where $\tau = |\sigma^{\tau_k} - \sigma^{\tau_m}| = O(h^{\tau_m+1})$ for some $0 \leq m \leq \ell$.

**Proof.** We first prove the results for the WENO-JS weights. For any $k$, we have
\begin{equation}
\tilde{\alpha}^{\tau_k} = \frac{\alpha^{\tau_k}}{\sigma^{\tau_k} + \epsilon_h} \frac{(\sigma^{\tau_k} + \epsilon_h)^\eta}{\sigma^{\tau_k} + \epsilon_h} \sum_{j=0}^\ell \frac{\alpha^{\tau_j}}{(\sigma^{\tau_j} + \epsilon_h)^\eta} = \frac{\alpha^{\tau_k}}{\sigma^{\tau_k} + \epsilon_h} \sum_{j=0}^\ell \frac{(\sigma^{\tau_k} + \epsilon_h)^\eta}{(\sigma^{\tau_j} + \epsilon_h)^\eta} \sum_{j=0}^\ell \frac{\alpha^{\tau_j} (1 + \frac{\sigma^{\tau_k} - \sigma^{\tau_j}}{\sigma^{\tau_j} + \epsilon_h})^\eta}{\sigma^{\tau_k} + \epsilon_h}.
\end{equation}

Now by (2.6),
\begin{equation}
\sigma^{\tau_k} + \epsilon_h = \begin{cases} \Theta(1), & \text{if } \epsilon_h = \epsilon, \\
\Theta(h^2), & \text{if } \epsilon_h = K h^2, \end{cases}
\end{equation}
and by Lemma 4.1, we have

\[ \sigma^{r_k} - \sigma^{r_j} = O(h^{\min(r_k,r_j)+1}). \]  

Hence

\[ \sum_{j=0}^{\ell} \alpha^{r_j} \left( 1 + \frac{\sigma^{r_k} - \sigma^{r_j}}{\sigma^{r_j} + \epsilon_h} \right) = \begin{cases} 1 + O(h^{r_0+1}), & \text{if } \epsilon_h = \epsilon, \\ 1 + O(h^{r_0+1}), & \text{if } \epsilon_h = K h^2. \end{cases} \]  

Combining this with (4.7) and recalling that \( r_0 \geq 2 \) shows that the result (4.5) holds for the WENO-JS weights.

Now for the WENO-Z weights, let \( \rho_{r,j} = \tau / (\sigma^{r_j} + \epsilon_h) \) and write

\[ \tilde{\alpha}^{r_k} = \frac{\alpha^{r_k}(1 + \rho_{r,k}^\eta)}{\sum_{j} \alpha^{r_j}(1 + \rho_{r,j}^\eta)} = \frac{\alpha^{r_k}(1 + \rho_{r,k}^\eta)}{1 + \sum_{j} \alpha^{r_j} \rho_{r,j}^\eta}. \]  

For any \( j \), since \( \tau = |\sigma^{r_k} - \sigma^{r_m}| = O(h^{r_m+1}) \),

\[ \rho_{r,j} = \frac{\tau}{\sigma^{r_j} + \epsilon_h} = \begin{cases} O(h^{r_m+1}), & \text{if } \epsilon_h = \epsilon, \\ O(h^{r_0-1}), & \text{if } \epsilon_h = K h^2, \end{cases} \]  

by (4.8)–(4.9). Since \( r_0 \geq 2 \), \( \rho_{r,j} \to 0 \) as \( h \to 0 \), and so

\[ \tilde{\alpha}^{r_k} \sim \alpha^{r_k}(1 + \rho_{r,k}^\eta) \left( 1 - \sum_{j} \alpha^{r_j} \rho_{r,j}^\eta \right) \]

\[ \sim \alpha^{r_k} \left( 1 + \rho_{r,k}^\eta - \sum_{j} \alpha^{r_j} \rho_{r,j}^\eta \right) \]

\[ = \alpha^{r_k} \left( 1 + \sum_{j} \alpha^{r_j} (\rho_{r,k}^\eta - \rho_{r,j}^\eta) \right). \]  

The mean value theorem shows that

\[ \rho_{r,k}^\eta - \rho_{r,j}^\eta = \eta \xi^{\eta-1} (\rho_{r,k} - \rho_{r,j}), \]  

for some \( \xi \) between \( \rho_{r,k} \) and \( \rho_{r,j} \). Furthermore, by (4.8)–(4.9) and (4.12), for all \( k \) and \( j \),

\[ \rho_{r,k} - \rho_{r,j} = \tau \left( \frac{1}{\sigma^{r_k} + \epsilon_h} - \frac{1}{\sigma^{r_j} + \epsilon_h} \right) \]

\[ = \tau \left( \frac{\sigma^{r_k} + \epsilon_h}{(\sigma^{r_k} + \epsilon_h)(\sigma^{r_j} + \epsilon_h)} \right) \]

\[ = \begin{cases} O(h^{r_m+r_0+2}), & \text{if } \epsilon_h = \epsilon, \\ O(h^{r_0+2}), & \text{if } \epsilon_h = K h^2. \end{cases} \]  

Combining (4.12)–(4.15) gives the conclusion (4.6).

4.3. Accuracy. We now present our results on the accuracy of the various WENO reconstructions when \( u \) is smooth. After this presentation, we provide a discussion of the results.
4.3.1. Standard WENO. For standard WENO, we generalize the results in [1] as follows.

**Theorem 4.3.** Let \( \eta \geq 1, \epsilon > 0, \) and \( K > 0. \) When \( u \) is smooth on \( S_0^r, r = 2s-1, \) \( s \geq 2, \) the standard WENO reconstruction \( R_r(x) \) is order \( r \) accurate at the point \( x^* \) defined in (3.2), using \( \epsilon_h = \epsilon \) or \( Kh^2 \) and either WENO-JS or WENO-Z weights.

**Proof.** We consider only the case of WENO-Z weights, since the case of WENO-JS weights is similar and can be found in [1]. By Lemma 4.1, \( \tau = O(h^{s+1}), \) so (4.6) is valid. We compute that

\[
R_r(x^*) - u(x^*) = \sum_j \tilde{\alpha}_j^*(P_j^s(x^*) - u(x^*))
\]

(4.16)

\[
= \sum_j \alpha_j^s(P_j^s(x^*) - u(x^*)) + \sum_j (\tilde{\alpha}_j^s - \alpha_j^s)(P_j^s(x^*) - u(x^*))
\]

\[
= (P_0^r(x^*) - u(x^*)) + \sum_j (\tilde{\alpha}_j^s - \alpha_j^s)(P_j^s(x^*) - u(x^*))
\]

\[
= O(h^r) + O(h^{(s-1)(\eta+1)})O(h^s),
\]

using (4.6) with \( r_j = r_m = r_0 = s. \)

4.3.2. WENO-AO\((r,s)\). The next theorem gives the accuracy of convergence of two-level WENO-AO\((r,s)\) given in (3.4).

**Theorem 4.4.** Let \( \eta \geq 1, \epsilon > 0, \) and \( K > 0. \) For \( r > s \geq 2, \) WENO-AO\((r,s)\) has order of accuracy \( \min(r, r_{\max}) \) on \( I_i \) if \( u \) is smooth on \( S_0^r, \) where for WENO-JS weights,

\[
r_{\max} = \begin{cases} 
2s + 1, & \text{if } \epsilon_h = \epsilon, \\
2s - 1, & \text{if } \epsilon_h = Kh^2,
\end{cases}
\]

and for WENO-Z weights,

\[
r_{\max} = r_{\max}(\eta) = \begin{cases} 
2s + 1 + (s+1)\eta, & \text{if } \epsilon_h = \epsilon, \\
2s - 1 + (s-1)\eta, & \text{if } \epsilon_h = Kh^2.
\end{cases}
\]

**Proof.** Because \( \alpha_0^r + \sum_j \alpha_j^s = \tilde{\alpha}_0^r + \sum_j \tilde{\alpha}_j^s = 1, \) we have on \( I_i \) that

\[
R^{r,s} - u = \frac{\tilde{\alpha}_0^r}{\alpha_0^s}[(P_0^r - u) - \sum_j \alpha_j^s(P_j^s - u)] + \sum_j \tilde{\alpha}_j^s(P_j^s - u)
\]

(4.19)

\[
= \frac{\tilde{\alpha}_0^r}{\alpha_0^s}(P_0^r - u) - \sum_j \left[\frac{\tilde{\alpha}_0^r - \alpha_0^s}{\alpha_0^s}\alpha_j^s - (\tilde{\alpha}_j^s - \alpha_j^s)\right](P_j^s - u)
\]

\[
= O(h^r) + \sum_j [O(\tilde{\alpha}_0^s - \alpha_0^s) + O(\tilde{\alpha}_j^s - \alpha_j^s)]O(h^s).
\]

Applying Theorem 4.2, we determine the value of \( r_{\max}. \)

4.3.3. WENO-AO\((r_\ell, r_{\ell-1}, \ldots, r_1, s)\). We can extend the above theorem to the multilevel WENO-AO\((r_\ell, r_{\ell-1}, \ldots, r_1, s)\) given in (3.7).

**Theorem 4.5.** Let \( \eta \geq 1, \epsilon > 0, K > 0, \) and \( \ell \geq 1. \) Let \( r_\ell > r_{\ell-1} > \cdots > r_1 > s \geq 2, \) and assume that \( u \) is smooth on \( S_0^{r_\ell}. \) Then WENO-AO\((r_\ell, \ldots, r_1, s)\) has order
of accuracy \( \min(r_\ell, r_{\max}) \) on \( I_i \), where \( r_{\max} \) is given by (4.17) for WENO-JS weights and (4.18) for WENO-Z weights when using a constant value for \( \eta \).

Moreover, if WENO-Z weights are used with variable \( \eta \), then the reconstruction \( U_i \) has order of accuracy \( r_\ell \) on \( I_i \), provided that

\[
(4.20) \quad r_k \leq r_{\max}(\eta_{0,k}), \quad \text{for all } 1 \leq k \leq \ell,
\]

and, for all \( 2 \leq k \leq \ell \),

\[
(4.21) \quad r_k \leq \begin{cases} 
  s + 1 + r_{k-1} + (r_{k-1} + 1)\eta_k, & \text{if } \epsilon_h = \epsilon, \\
  s - 1 + r_{k-1} + (r_{k-1} - 1)\eta_k, & \text{if } \epsilon_h = Kh^2,
\end{cases}
\]

\[
(4.22) \quad r_k \leq \begin{cases} 
  3s + 2 + (r_{k-1} + 1)\eta_k + (s + 1)\eta_{\ell-1}, & \text{if } \epsilon_h = \epsilon, \\
  3s - 2 + (r_{k-1} - 1)\eta_k + (s - 1)\eta_{\ell-1}, & \text{if } \epsilon_h = Kh^2,
\end{cases}
\]

where \( r_0 = s \) and \( \eta_1 = \eta_{0,1} \).

**Proof.** For fixed \( s \geq 2 \), the proof proceeds by induction on \( \ell \geq 1 \). The result holds for \( \ell = 1 \) by Theorem 4.4. Assume the result holds for \( \ell - 1 \geq 1 \). We write the argument for variable \( \eta \), since we can simply fix the value for the first part of the theorem. We compute on \( I_i \) that

\[
R^{s, r_{\ell-1}, \ldots, r_1, s}_u = \frac{\tilde{\gamma}^{r_{\ell}, s}}{\gamma^{r_{\ell}, s}} [(R^{r_{\ell}, s} - u) - \gamma^{r_{\ell-1}, \ldots, r_1, s}(R^{r_{\ell-1}, \ldots, r_1, s} - u)] \\
+ \frac{\tilde{\gamma}^{r_{\ell-1}, \ldots, r_1, s}}{\gamma^{r_{\ell-1}, \ldots, r_1, s}} (R^{r_{\ell-1}, \ldots, r_1, s} - u) \\
= \mathcal{O}(h_{\min}(r_\ell, r_{\max}(\eta_{\ell}))) \\
= \mathcal{O}(h_{\min}(r_\ell, r_{\max}(\eta_{\ell}))) + \mathcal{O}(\tilde{\gamma}^{r_{\ell}, s} - \gamma^{r_{\ell}, s}) \\
+ \mathcal{O}(\tilde{\gamma}^{r_{\ell-1}, \ldots, r_1, s} - \gamma^{r_{\ell-1}, \ldots, r_1, s}) \mathcal{O}(h_{\min}(r_{\ell-1}, r_{\max}(\eta_{\ell-1}))),
\]

using Theorem 4.4 and induction. If the perturbation of the linear weights is written as \( \mathcal{O}(h^w_{\epsilon}) \), then we have

\[
(4.24) \quad R^{s, r_{\ell-1}, \ldots, r_1, s}_u = \mathcal{O}(h_{\min}(r_\ell, r_{\max}(\eta_{\ell}))) \\
= \mathcal{O}(h_{\min}(r_\ell, r_{\max}(\eta_{\ell}))) + \mathcal{O}(\tilde{\gamma}^{r_{\ell}, s} - \gamma^{r_{\ell}, s}) \\
+ \mathcal{O}(\tilde{\gamma}^{r_{\ell-1}, \ldots, r_1, s} - \gamma^{r_{\ell-1}, \ldots, r_1, s}) \mathcal{O}(h_{\min}(r_{\ell-1}, r_{\max}(\eta_{\ell-1}))),
\]

where \( w_\ell \) is given in Theorem 4.2 with \( r_m = r_{\ell-1}, r_0 = s \), and \( \eta = \eta_{\ell} \) as

\[
w_\ell = \begin{cases} 
  s \pm 1 & \text{for WENO-JS weights,} \\
  s + 1 + (r_{\ell-1} \pm 1)\eta_{\ell} & \text{for WENO-Z weights,}
\end{cases}
\]

respectively for \( \epsilon_h = \epsilon \) (positive sign) and \( \epsilon_h = Kh^2 \) (negative sign).

For the first part of the theorem, WENO-Z weights use a constant \( \eta \), and so both types of weights lead to \( w_\ell + r_{\ell-1} \geq w_\ell + s \geq r_{\max}(\eta) \). Thus,

\[
\min(r_\ell, r_{\max}(\eta), w_\ell + r_{\ell-1}, w_\ell + r_{\max}(\eta)) = \min(r_\ell, r_{\max}(\eta)).
\]

For the second part of the theorem, i.e., when WENO-Z weights are used with variable \( \eta \), note that

\[
w_\ell + r_{\ell-1} = s \pm 1 + (r_{\ell-1} \pm 1)\eta_{\ell} + r_{\ell-1} \geq r_\ell,
\]
by \((4.21)\). Moreover, \((4.22)\) shows that
\[
w_{\ell} + r_{\max}(\eta_{\ell-1}) = 3s \pm 2 + (r_{\ell-1} \pm 1)\eta_{\ell} + (s \pm 1)\eta_{\ell-1} \geq r_{\ell},
\]
so, with \((4.20)\)
\[
\min(r_{\ell}, r_{\max}(\eta_{0,\ell}), w_{\ell} + r_{\ell-1}, w_{\ell} + r_{\max}(\eta_{\ell-1})) = r_{\ell},
\]
and the proof is complete.

\subsection*{4.3.4. WENO-AO\((r_{\ell}, r_{\ell-1}, \ldots, r_{0})\).}

The following theorem discusses the new reconstruction \((3.12)\).

\begin{theorem}
Let \(\epsilon > 0\), \(K > 0\), and \(\ell \geq 1\). Let \(r_{\ell} > r_{\ell-1} > \cdots > r_{0} \geq 2\), and assume that \(u\) is smooth on \(S_{0}^{r_{\ell}}\). Let WENO-JS weights or WENO-Z weights be used with parameter \(\eta_{k} \geq 1\) on level \(r_{k}\), and define \(r_{\max,0} = r_{0}\) and, for \(1 \leq k \leq \ell\),
\begin{equation}
(4.25) \quad r_{\max, k} = \min(r_{k-1}, r_{\max, k-1}) + \begin{cases} r_{0} \pm 1 & \text{for WENO-JS weights}, \\ r_{0}(\eta_{k} + 1) & \text{for WENO-Z weights}, \end{cases}
\end{equation}
respectively for \(\epsilon_{k} = \epsilon + \text{sign}\) and \(\epsilon_{k} = K\eta_{k}^{2} - \text{sign}\). Then WENO-AO\((r_{\ell}, \ldots, r_{0})\) has order of accuracy \(\min(r_{\ell}, r_{\max,\ell})\) on \(I_{\ell}\).
Moreover, WENO-AO\((r_{\ell}, \ldots, r_{0})\) has order of accuracy \(r_{\ell}\) on \(I_{\ell}\) if, for all \(1 \leq k \leq \ell\),
\begin{equation}
(4.26) \quad r_{k} \leq \begin{cases} r_{k-1} + r_{0} \pm 1 & \text{for WENO-JS weights}, \\ r_{k-1} + (r_{0} \pm 1)(\eta_{k} + 1) & \text{for WENO-Z weights}. \end{cases}
\end{equation}
\end{theorem}

\begin{proof}
The proof is similar to the inductive proof of Theorem 4.5. The result holds for \(\ell = 1\) by Theorem 4.4, so assume the result holds for \(\ell - 1\). We compute on \(I_{\ell}\) that
\begin{equation}
(4.27) \quad R^{r_{\ell}, \ldots, r_{0}} - u = \frac{\tilde{\alpha}_{0}^{r_{\ell}}}{\alpha_{0}^{r_{\ell}}} [(P_{\ell}^{r_{\ell}} - u) - \left(\sum_{j} \alpha_{j}^{r_{\ell}}\right)(R^{r_{\ell-1}, \ldots, r_{0}} - u)] + \left(\sum_{j} \tilde{\alpha}_{j}^{r_{\ell}}\right)(R^{r_{\ell-1}, \ldots, r_{0}} - u).
\end{equation}
By Theorem 4.4 and induction, we have
\[
R^{r_{\ell}, \ldots, r_{0}} - u = O(h^{r_{\ell}}) + \left[O(\tilde{\alpha}_{0}^{r_{\ell}} - \alpha_{0}^{r_{\ell}}) + \sum_{j} O(\tilde{\alpha}_{j}^{r_{\ell}} - \alpha_{j}^{r_{\ell}})\right]O(h^{\min(r_{\ell}, r_{\max,\ell-1})}).
\]
The perturbation of nonlinear weights is given by Theorem 4.2 with \(r_{m} = r_{0}\), and the main result follows. The result \((4.26)\) is given by requiring \(r_{k} \leq r_{\max, k}\) for all \(k\). \end{proof}

\subsection*{4.3.5. Discussion.}
Standard WENO reconstructions have a simple convergence theory. They give the optimal high-order convergence rate whenever \(u\) is smooth.

The two-level WENO-AO\((r, s)\) achieves the optimal convergence \(O(h^{r})\) when the base level \(s\) is sufficiently high. In terms of the gap \(r - s\) between levels, one needs
\begin{equation}
(4.28) \quad r - s \leq \begin{cases} s \pm 1 & \text{for WENO-JS weights}, \\ (s \pm 1)(\eta + 1) & \text{for WENO-Z weights}, \end{cases}
\end{equation}
respectively for \( \epsilon_h = \epsilon \) (+ sign) and \( \epsilon_h = Kh^2 \) (- sign). The WENO-Z weights are interesting in that one can adjust the value of \( \eta \) to reduce the constraint.

For the multilevel WENO reconstructions with adaptive order, the weights used has a marked effect on the results. The WENO-JS weights have a simple convergence theory. The optimal convergence \( \mathcal{O}(h^r) \) is attained by WENO-AO\(_s\)(\( r_\ell, \ldots, r_1, s \)) with base level \( s \) when

\[
r_\ell - s \leq s \pm 1,
\]

but the new WENO-AO\(_s\)(\( r_\ell, \ldots, r_1, r_0 \)) requires only that

\[
r_k - r_{k-1} \leq r_0 \pm 1, \quad \forall 1 \leq k \leq \ell.
\]

The condition for WENO-AO\(_s\) is that the largest gap \( r_\ell - s \) must be bounded by \( s \pm 1 \), independent of the intermediate levels. In contrast, the new WENO-AO merely requires that each intermediate gap be bounded by this number, i.e., \( r_0 \pm 1 \).

Obtaining optimal accuracy with WENO-Z weights is a much more complex proposition. WENO-AO\(_s\)(\( r_\ell, \ldots, r_1, s \)) has the three conditions (4.20)–(4.22). The first condition is

\[
r_k - s \leq (s \pm 1)(\eta_{0,k} + 1) \quad \forall 1 \leq k \leq \ell.
\]

That is, each two-level approximation must be accurate, and then the gaps in the levels must satisfy the two relatively relaxed bounds (4.21)–(4.22). The new WENO-AO\(_s\)(\( r_\ell, \ldots, r_1, r_0 \)) has only the condition (4.26), i.e.,

\[
r_k - r_{k-1} \leq (r_0 \pm 1)(\eta_k \pm 1) \quad \forall 1 \leq k \leq \ell,
\]

This condition is worse than (4.21)–(4.22), but the very stringent condition (4.31) is removed.

A careful choice of \( \eta \)'s can recover the full accuracy when using WENO-Z weights for any chosen approximation levels. As (4.32) shows, the new WENO-AO can use a bounded set of \( \eta \)'s, whatever value for \( r_\ell \) is taken. The condition (4.31) for WENO-AO\(_s\) requires very large values of \( \eta \) when \( r_\ell \) is taken very large.

While large values of \( \eta \) improve the convergence rates, they do so by strongly biasing the values of the nonlinear weights to that of the linear ones. This has a tendency to diminish the essentially nonoscillatory property of WENO schemes for solving problems with shocks. This was noticed in [2]: the authors remarked that WENO-JS weights were more stable, while WENO-Z weights gave better convergence results in the smooth case.

5. Accuracy analysis in the discontinuous case. We now consider the case when \( u \) is not smooth over the big stencil, but \( u \) is smooth on some of the smaller stencils. We consider only the case that \( u \) is smooth on a stencil or has a jump discontinuity somewhere in its interior. That is, we do not consider the intermediate case where \( u \) is continuous but pertinent derivatives are discontinuous, nor the case of multiple discontinuities, because we are interested in reconstructions involving a single shock discontinuity.

5.1. Smoothness indicators in the discontinuous case. In general, when there is an actual discontinuity, it is obvious that the smoothness indicator is \( \mathcal{O}(1) \), as noted in (2.6). However, it is far from obvious that \( \sigma = \Theta(1) \), and this is in general not true, as we will see in Example 5.3 below. The result \( \sigma = \Theta(1) \) holds for some particular sequences of grids.
**Definition 5.1.** Let \( h > 0 \) and \( x_n^h \) be the gridpoints with maximal spacing \( h \). For \( x_* \) fixed, let \( m \) be defined so \( x_m^h < x_* < x_{m+1}^h \). We say that \( x_* \) is bounded away from the gridpoints as \( h \to 0 \) if there exists a constant \( c_* \in (0, 1) \) such that

\[
0 < c_* \Delta x_m \leq x_* - x_m^h \quad \text{and} \quad 0 < c_* \Delta x_m \leq x_{m+1}^h - x_*
\]

for all \( h \). We also say that the grids are bounded away from \( x_* \) as \( h \to 0 \).

**Lemma 5.2.** Let cell \( I_i \) and the stencil \( S^r \supseteq I_i \), where \( r \geq 2 \), be given. Assume that \( u \) is smooth except for a jump discontinuity at \( x_* \in I_m \in S^r \). If \( x_* \) is bounded away from the gridpoints as \( h \to 0 \), then the smoothness indicator

\[
\sigma^r = \Theta(1) \quad \text{as} \quad h \to 0.
\]

**Proof.** Define the reference coordinate \( \xi = (x - x_{i+\frac{1}{2}})/h \). The stencil polynomial \( P^r \) can be written as

\[
P^r(x) = \sum_{j=0}^{r-1} a_j \left( \frac{x - x_{i+\frac{1}{2}}}{h} \right)^j = \sum_{j=0}^{r-1} a_j \xi^j = \hat{P}^r(\xi),
\]

where \( a_j \) dependents only on the cell averages. By (2.4), we compute

\[
\sigma^r = \sum_{\ell=1}^{r-1} \int_{I_i} (\Delta x_j)^{2\ell-1} \left( \frac{d^{2\ell} P^r(x)}{dx^{2\ell}} \right)^2 dx = \sum_{\ell=1}^{r-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{d^{2\ell} \hat{P}^r(\xi)}{d\xi^{2\ell}} \right)^2 d\xi.
\]

Hence \( \sigma^r \) depends only on the cell averages. As \( h \to 0 \), if \( \sigma^r \to 0 \), then all the derivatives of \( \hat{P}^r \) converge to 0 on \([\frac{-1}{2}, \frac{1}{2}]\), i.e., \( \hat{P}^r \) converges to some constant, and the same is true of \( P^r \). However, as \( h \to 0 \),

\[
\hat{u}_j = \mathcal{O}(h) + \begin{cases} u(x_*^+), & \text{if } j < m, \\ c_* u(x_*^-) + (1 - c_*) u(x_*^+), & \text{if } j = m, \\ u(x_*^+), & \text{if } j > m. \end{cases}
\]

Since \( r \geq 2 \), the cell averages converge to at least two distinct values. So \( P^r \) cannot converge to a constant, which is a contradiction. Thus there exists some positive constant \( C_1 \) such that \( \sigma^r \geq C_1 \). Clearly \( P^r \) is smooth, and all the cell averages are bounded, so \( \sigma^r \leq C_2 \) holds for some positive \( C_2 \). \( \square \)

We remark that for a specific stencil (actually a specific sequence of stencils), the proof shows that the smoothness indicator \( \sigma^r \to 0 \) only if \( x_* \) is near the endpoints of the stencil. That is, \( x_* \) must be in the left-most or right-most cell of the stencil. This is the only case in which as \( h \to 0 \), if we allowed \( c_* \to 0 \), then we would have only a single value for the cell averages arising in (5.3). However, WENO reconstruction involves a combination of substencils. So if the discontinuity is near the gridpoints anywhere in the big stencil, some small substencil will have this endpoint property. We thus make Definition 5.1 apply to all the gridpoints.

The above lemma does not hold for all sequences of grids, as shown in the next example, where \( c_* = h \).

**Example 5.3.** Given cell \( I_i = [h - h^2, 2h - h^2] \) and

\[
(5.4) \quad u(x) = H(-x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases}
\]
where \( H \) is the Heaviside function, consider the stencil \( S^2_0 = \{[-h^2, h - h^2], I_i\} \), for which the average of \( u \) on each cell is \( h \) and 0, respectively. The stencil polynomial is

\[
P_0^j(x) = \frac{3h}{2} - h^2 - x,
\]

and the smoothness indicator (2.4) is

\[
\sigma^2 = h^2 = \Theta(h^2) \neq \Theta(1).
\]

The literature is fraught with the belief that \( \sigma \to 0 \) as \( h \to 0 \) when there is a discontinuity (e.g., in [1], this is assumed as a hypothesis, and in [8], this belief is stated as being obvious).

### 5.2. WENO approximation on grids bounded away from the discontinuity.

In [1], the authors showed that in the discontinuous case when the smoothness indicator \( \sigma \to 0 \), WENO approximations are expected to converge only if \( \varepsilon_h = O(h) \), so we only consider the case \( \varepsilon_h = K h^2 \). The next theorem gives the magnitude of WENO weights as \( h \to 0 \). The results for WENO-JS weights appear in [1] for standard WENO and in [8] for compact CWENO3.

**Theorem 5.4.** Let \( \eta \geq 1 \), \( K > 0 \), and \( \varepsilon_h = K h^2 \). Given cell \( I_i \), let \( S^r_j \supset I_i \) be a stencil of size \( r_j \geq 2 \), for \( j = 0, 1, \ldots, \ell \). Assume that \( r_0 \leq r_1 \leq \cdots \leq r_\ell \). Let \( \hat{\alpha}^r_j \) be the positive linear weights such that \( \sum_j \alpha^r_j = 1 \), and let \( \sigma^r_j \) be the corresponding smoothness indicators. If \( u \) is smooth except for a single discontinuity, and if \( u \) is smooth on at least one stencil, then for grids bounded away from the discontinuity,

\[
\alpha^r_j = \begin{cases} 
\Theta(1), & \text{if } u \text{ is smooth on } S^r_j, \\
\Theta(h^{2\eta}), & \text{if } u \text{ has a jump discontinuity on } S^r_j,
\end{cases}
\]

for all \( j = 0, 1, \ldots, \ell \), for both WENO-JS and WENO-Z weights provided \( \tau = \Theta(1) \).

We remark that the WENO-Z weights defined in (2.9) for standard WENO require \( r \geq 3 \). When \( r \) is odd, \( \tau = \Theta(1) \), since \( \tau = |\sigma_{-k} - \sigma_k|, \ k = \left\lfloor \frac{r}{2} \right\rfloor \), compares the leftmost and rightmost stencils, only one of which contains the discontinuity. It is not clear whether \( \tau = \Theta(1) \) when \( r \) is even in (2.9).

**Proof.** First consider the WENO-JS weights. By (2.6), Lemmas 4.1 and 5.2, for any \( j \neq k \),

\[
\frac{\sigma^{r_j} - \sigma^{r_k}}{\sigma^{r_k} + \varepsilon_h} = \begin{cases} 
\Theta(h^{\min(r_j, r_k) - 1}), & \text{if } u \text{ is smooth on } S^{r_j} \text{ and } S^{r_k}, \\
\Theta(1), & \text{if } u \text{ is smooth on } S^{r_j}, \text{ but jumps on } S^{r_k}, \\
\Theta(h^{-2}), & \text{if } u \text{ jumps on } S^{r_j}, \text{ but is smooth on } S^{r_k}, \\
\Theta(1), & \text{if } u \text{ jumps on } S^{r_j} \text{ and } S^{r_k}.
\end{cases}
\]

Hence we obtain

\[
\sum_{k=0}^{m} \alpha^{r_k} \left( 1 + \frac{\sigma^{r_j} - \sigma^{r_k}}{\sigma^{r_k} + \varepsilon_h} \right)^\eta = \begin{cases} 
\Theta(1), & \text{if } u \text{ is smooth on } S^{r_j}, \\
\Theta(h^{-2\eta}), & \text{if } u \text{ jumps on } S^{r_j},
\end{cases}
\]

and so (4.7) and (5.6) imply the result (5.5).

For the WENO-Z weights, since \( \tau = \Theta(1) \), (4.8) shows

\[
\rho^{r_j} = \frac{\tau}{\sigma^{r_j} + \varepsilon_h} = \begin{cases} 
\Theta(h^{-2}), & \text{if } u \text{ is smooth on } S^{r_j}, \\
\Theta(1), & \text{if } u \text{ jumps on } S^{r_j}.
\end{cases}
\]
Thus the denominator in (4.11) is dominated by $\Theta(h^{-2})$, and the result follows.

We present in the next theorems the accuracy of the WENO reconstructions. The first theorem is a generalization of a result in [1].

### 5.2.1. Standard WENO and WENO-AO($r, s$).

**Theorem 5.5.** Let $K > 0$ and $\epsilon_h = Kh^2$. Given cell $I_i$, let $u$ be smooth except for a jump discontinuity $x_*, I_{m} \neq I_i$. Assume that the grids are bounded away from the discontinuity, and that WENO-JS weights or WENO-Z weights are used, where in the latter case $\tau = \Theta(1)$. For the standard WENO reconstruction $R_r$, $r = 2s - 1$, $s \geq 2$, for the point $x^*$ in (3.1), where $I_m \in S_0^{2s-1}$,

$$|R_r(x^*) - u(x^*)| = \Theta(h^r) \quad \text{if} \quad \eta \geq s/2.$$  \hspace{1cm} (5.8)

For the WENO-AO($r, s$) reconstruction $R^{r,s}$, $r > s \geq 2$, where $I_m \in S_0^s$ and $I_i \in S_j^s \subseteq S_0^s$, for all $j$, on $I_i$,

$$|R^{r,s}(x) - u(x)| = \Theta(h^r) \quad \forall x \in I_i \quad \text{if} \quad \eta \geq s/2.$$  \hspace{1cm} (5.9)

**Proof.** For either weights, we have

$$|R_r(x^*) - u(x^*)| = \left| \sum_j \tilde{\alpha}_j^s (P_j^s(x^*) - u(x^*)) \right|$$

$$\leq \sum_{u \text{ discontinuous on } S_j^s} \tilde{\alpha}_j^s |P_j^s(x^*) - u(x^*)| + \sum_{u \text{ smooth on } S_j^s} \tilde{\alpha}_j^s |P_j^s(x^*) - u(x^*)|$$

$$= \Theta(h^{2\eta}) \Theta(1) + \Theta(1) \Theta(h^r).$$

by Theorem 5.4, and (5.8) follows. Again using Theorem 5.4, we also compute,

$$|R^{r,s}(x) - u(x)|$$

$$\leq \left| \frac{\tilde{\alpha}_j^r}{\tilde{\alpha}_j^s} (P_j^r(x) - u(x)) - \sum_j \alpha_j^s (P_j^s(x) - u(x)) \right| + \sum_j \tilde{\alpha}_j^s |P_j^s(x) - u(x)|$$

$$\leq \Theta(\tilde{\alpha}_j^s) \Theta(1) + \Theta(h^s) + \left\{ \sum_{u \text{ discontinuous on } S_j^s} \Theta(\tilde{\alpha}_j^s) \Theta(1) + \sum_{u \text{ smooth on } S_j^s} \Theta(\tilde{\alpha}_j^s) \Theta(h^r) \right\}$$

$$= \Theta(h^{2\eta}) \Theta(1) + \Theta(h^r) + \left\{ \sum_{u \text{ discontinuous on } S_j^s} \Theta(h^{2\eta}) \Theta(1) + \sum_{u \text{ smooth on } S_j^s} \Theta(1) \Theta(h^s) \right\}.$$  

Therefore we conclude the result (5.9). 

The example below shows that when there is a jump discontinuity bounded away from the gridpoint on the big stencil, the standard WENO reconstruction $R_r$ and the WENO-AO($r, s$) reconstruction may not drop to order $s$ when $\eta < s/2$. That is, the requirement that $\eta \geq s/2$ is sharp.

**Example 5.6.** Given cell $I_i = \left[ \frac{h}{2}, \frac{3h}{2} \right]$ and $u$ defined by (5.4), consider the stencil $S_0^5 = \left\{ \left[ -\frac{3h}{2}, -\frac{h}{2} \right], \left[ -\frac{h}{2}, \frac{h}{2} \right], I_i, \left[ \frac{3h}{2}, \frac{5h}{2} \right], \left[ \frac{5h}{2}, \frac{7h}{2} \right] \right\}$. The average of $u$ on each cell is 1, 1/2, 0, 0, and 0, respectively. Consider the standard WENO reconstruction $R_3$ and the WENO-AO($5, 3$) reconstruction with $\epsilon_h = h^2$. The stencil polynomials are

$$P_{-1}^5(x) = \frac{1}{2} - \frac{x}{2h}, \quad P_0^5 = \frac{23}{48} - \frac{3x}{4h} + \frac{x^2}{4h^2}, \quad P_1^5 = 0,$$

$$P_0^5(x) = \frac{317}{640} - \frac{17x}{24h} + \frac{x^2}{16h^2} + \frac{x^3}{6h^3} - \frac{x^4}{24h^4}.$$
Therefore, the errors at \( x = \frac{h}{2} \) (the leftmost point of \( I_i \)) are

\[
P^3_1(\frac{h}{2}) - u(\frac{h}{2}) = \frac{1}{4} = \Theta(1), \quad P^3_0(\frac{h}{2}) - u(\frac{h}{2}) = \frac{1}{6} = \Theta(1), \quad P^3_1(\frac{h}{2}) - u(\frac{h}{2}) = 0, \quad P^3_2(\frac{h}{2}) - u(\frac{h}{2}) = \frac{7}{40} = \Theta(1),
\]

and the smoothness indicators are

\[
\sigma^2_1 = \frac{1}{4} = \Theta(1), \quad \sigma^3_0 = \frac{1}{3} = \Theta(1), \quad \sigma^3_1 = 0, \quad \sigma^3_2 = 30593 \quad 20160 = \Theta(1).
\]

By Theorem 5.4, since \( \tau = \Theta(1) \), for both weights, we have

\[
\tilde{\alpha}^3_{-1} = \Theta(h^{2q}), \quad \tilde{\alpha}^3_0 = \Theta(h^{2q}), \quad \tilde{\alpha}^3_0 = \Theta(h^{2q}).
\]

So the error of the reconstruction \( R_3 \) at \( x = \frac{h}{2} \) is

\[
R_3(\frac{h}{2}) - u(\frac{h}{2}) = \sum_{j=1}^{1} \tilde{\alpha}^3_j [P^3_j(\frac{h}{2}) - u(\frac{h}{2})] = \mathcal{O}(h^{2q}) \leq \Theta(h^{2q}).
\]

On the other hand, for \( R^{5,3} \) at \( x = \frac{h}{2} \),

\[
R^{5,3}(\frac{h}{2}) - u(\frac{h}{2}) = \tilde{\alpha}^5 \tilde{\alpha}^3_0 \left\{ [P^5_0(\frac{h}{2}) - u(\frac{h}{2})] - \sum_{j=1}^{1} \tilde{\alpha}^3_j [P^3_j(\frac{h}{2}) - u(\frac{h}{2})] \right\} \\
+ \sum_{j=-1}^{1} \tilde{\alpha}^3_j [P^3_j(\frac{h}{2}) - u(\frac{h}{2})] = \mathcal{O}(h^{2q}) \leq \Theta(h^{2q}).
\]

Numerical results in Table 5.1 show that we can achieve \( \Theta(h^{2q}) \) convergence for \( R_3 \) and WENO-AO(5,3), for \( q = 1, 1.5, 2, 3 \). When \( \eta = 1 \), both of the reconstructions are only second order accurate instead of third for either WENO-JS and WENO-Z weights. We used the sequence of grid spacings \( \{h_n = 2^{-n}\}_{n=0}^{\infty} \) and \( \alpha_0^5 = 0.85 \), \( \alpha_0^3 = 0.05 \) for these results.

### 5.2.2. Multilevel WENO-AO\(_s\)(\( r_\ell, \ldots, r_1, s \))

We have the following result for the multilevel WENO-AO\(_s\)(\( r_\ell, \ldots, r_0, s \)) reconstructions.

**Theorem 5.7.** Let \( K > 0 \) and \( \epsilon_h = Kh^2 \). Let cell \( I_i \) be given, \( \ell \geq 2 \), \( r_\ell > r_{\ell-1} > \cdots > r_0 = s \geq 2 \), and \( I_i \in S_{i_0}^{r_0} \subseteq S_{i_1}^{r_1} \subseteq \cdots \subseteq S_{i_n}^{r_n} \), for all \( j \). Let \( u \) be smooth except for a jump discontinuity at \( x_* \in I_{m} \in S_{i_n}^{r_n} \), \( m \neq i \). Assume that \( x_* \in S_{i_n}^{r_n} \) but \( x_* \notin S_{j}^{r_j} \), \( 0 \leq n \leq \ell \) (\( j = 0 \) if \( r_n \geq 1 \)) and that WENO-JS weights or WENO-Z weights are used with variable \( \eta_k \). Then for the WENO-AO\(_s\)(\( r_\ell, \ldots, r_1, s \)) reconstruction, the following hold on grids bounded away from the discontinuity.

1. If \( n = \ell - 1 \), let \( q = \min(r_{\ell-1}, r_{\max}(\eta_{\ell-1})) \), where \( r_{\max} \) is given in Theorem 4.4. If \( \eta_* \geq (q - s)/2 \), \( \eta_{0,\ell} \geq s/2 \), and (4.20)–(4.22) hold with \( \ell \) replaced by \( \ell - 1 \), then on \( I_i \),

\[
|R^r_{s} r_{\ell}, \ldots, r_{1, s}(x) - u(x)| = \mathcal{O}(h^q) \quad \forall x \in I_i.
\]

2. If \( n < \ell - 1 \) and \( \eta_{0,k} \geq s/2 \), \( 1 \leq k \leq \ell \), then on \( I_i \),

\[
|R^r_{s} r_{\ell}, \ldots, r_{1, s}(x) - u(x)| = \mathcal{O}(h^q) \quad \forall x \in I_i.
\]
Table 5.1: Example 5.6, Standard WENO $R_3$ and WENO-AO(5, 3) error and convergence rate at $x = h/2$. The convergence rates are indeed $\Theta(h^{2q})$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\eta = 1$</th>
<th>$\eta = 1.5$</th>
<th>$\eta = 2$</th>
<th>$\eta = 3$</th>
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<td></td>
<td>error</td>
<td>order</td>
<td>error</td>
<td>order</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.45E-3</td>
<td>1.97</td>
<td>4.26E-5</td>
<td>2.99</td>
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<td>1.99</td>
<td>5.34E-6</td>
<td>3.00</td>
</tr>
<tr>
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<td>8.15E-5</td>
<td>2.00</td>
<td>6.67E-7</td>
<td>3.00</td>
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<tr>
<td>8</td>
<td>2.29E-5</td>
<td>2.00</td>
<td>8.34E-8</td>
<td>3.00</td>
</tr>
<tr>
<td>$\eta$</td>
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<td>1.93</td>
<td>9.60E-5</td>
<td>2.99</td>
</tr>
<tr>
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<td>7.90E-4</td>
<td>1.98</td>
<td>1.20E-5</td>
<td>3.00</td>
</tr>
<tr>
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<td>2.00</td>
<td>1.50E-6</td>
<td>3.00</td>
</tr>
<tr>
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<td>4.96E-5</td>
<td>2.00</td>
<td>1.88E-7</td>
<td>3.00</td>
</tr>
<tr>
<td>$\eta$</td>
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<td>1.98</td>
<td>1.72E-5</td>
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<td>2.69E-7</td>
<td>3.00</td>
</tr>
<tr>
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<td>1.35E-5</td>
<td>2.00</td>
<td>3.36E-8</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Note that WENO-AO($r_t, \ldots, r_1, s$) drops to the base order $s$ when $u$ has a jump discontinuity on the two biggest stencils.

**Proof.** If $n = \ell - 1$, then $\tau = |\sigma^{r_t} - \sigma^{r_{\ell-1}}| = \Theta(1)$ by (2.6) and Lemma 5.2. By Theorem 5.4, $\tilde{\gamma}^{r_t:s}(x) = \Theta(h^{2q})$ and $\tilde{\gamma}^{r_{\ell-1}:r_1:s}(x) = \Theta(1)$. Following (4.23) and using Theorems 4.5 and 5.5,

$$
| R^{r_t, \ldots, r_1:s}(x) - u(x) |
\leq \frac{\tilde{\gamma}^{r_t:s} R^{r_t:s}(x) - u(x)}{	ilde{\gamma}^{r_t:s} R^{r_t:s}(x) - u(x)}
\leq \frac{\Theta(\tilde{\gamma}^{r_t:s}) [O(h^s) + O(h^q)] + \Theta(\tilde{\gamma}^{r_{\ell-1}:r_1:s}) O(h^q)}{\Theta(h^{2q})}.
$$

(5.12)

Since $q \geq s$, we conclude (5.10).

When $0 \leq n < \ell - 1$, $u$ is smooth neither on $S^{r_t}_0$ nor on $S^{r_{\ell-1}}_0$, so Lemma 5.2 shows $\sigma^{r_t} = \Theta(1)$ and $\sigma^{r_{\ell-1}} = \Theta(1)$. Hence, following (3.8), (3.11) and (3.9), $\tilde{\gamma}^{r_t:s}(x)$ and $\tilde{\gamma}^{r_{\ell-1}:r_1:s}(x)$ is at least order $s$ accurate when $\eta_{0,k} \geq s/2, 1 \leq k \leq \ell$, then an argument similar to (5.12) shows (5.11). □

### 5.2.3. The new multilevel WENO-AO($r_t, \ldots, r_0$).

When $u$ is smooth only on some substencils of $S^{r_t}_0$, we have the following result for the new reconstruction.

**Theorem 5.8.** Let $K > 0$ and $\epsilon_h = Kh^2$. Let cell $I_i$ be given, $\ell \geq 2$, $r_t > r_{\ell-1} > \cdots > r_0 \geq 2$, and $I_i \in S^{r_0}_j \subseteq S^{r_1}_0 \subseteq \cdots \subseteq S^{r_t}_0$, for all $j$. Let $u$ be smooth...
except for a jump discontinuity at \( x_i \in I_m \in S_0^\ell \), \( m \neq i \). Assume that \( x_i \in S_0^{n+1} \) but \( x_i \notin S_n^\ell \), \( 0 \leq n < \ell \) (\( j = 0 \) if \( r_n \geq 1 \)) and that WENO-JS weights or WENO-Z weights are used with variable \( \eta_k \). Then on grids bounded away from the discontinuity, the WENO-AO\((r_t, \ldots, r_0)\) reconstruction satisfies on \( I_i \),

\[
|R^{t \ldots 0}(x) - u(x)| = O(h^p) \quad \text{if} \quad \eta_k \geq p/2, \; n + 1 \leq k \leq \ell,
\]

where \( p = \min(r_n, r_{\max,n}) \) with \( r_{\max,n} \) being given in \((4.25)\). Moreover, if \( r_k \leq r_{\max,k} \) for each \( 1 \leq k \leq n \) and \( q > 0 \) is fixed, then on \( I_i \),

\[
|R^{t \ldots 0}(x) - u(x)| = O(h^{\min(q, r_n)}) \quad \text{if} \quad \eta_k \geq q/2, \; n + 1 \leq k \leq \ell.
\]

Thus, if \( q = r_{\ell-1} \), the reconstruction drops to the best order possible. A smaller value of \( q \) may be chosen to keep the collection of \( \eta_k \) from becoming too large.

**Proof.** Following \((4.27)\),

\[
|R^{t \ldots 0}(x) - u(x)| \\
\leq \frac{\tilde{\alpha}_0^{r_t}}{\alpha_0^{r_t}} |R_0^{t \ldots 0}(x) - u(x)| + \left( \sum_j \alpha_j^{r_0} \right) |R^{t-1 \ldots 0}(x) - u(x)| \\
+ \left( \sum_j \tilde{\alpha}_j^{r_0} \right) |R^{t-1 \ldots 0}(x) - u(x)|.
\]

We will prove the result by induction on \( \ell \geq n + 1 \). When \( \ell = n + 1 \), by Theorems 4.6 and 5.4, we have

\[
|R^{t \ldots 0}(x) - u(x)| \\
\leq \Theta(\alpha_0^{r_0}) [O(1) + O(h^{\min(r_{\ell-1}, r_{\max, \ell-1})})] + \left( \sum_j \Theta(\tilde{\alpha}_j^{r_0}) \right) O(h^{\min(r_{\ell-1}, r_{\max, \ell-1})}) \\
= \Theta(h^{2\eta_0}) \cdot O(1) + \Theta(1) \cdot O(h^p).
\]

So the result holds when \( \ell = n + 1 \). Assume by induction that the result holds for some \( \ell - 1 \geq n + 1 \). Then \( |R^{t-1 \ldots 0}(x) - u(x)| = O(h^p) \), so by an argument similar to \((5.16)\), we conclude the result \((5.13)\) holds for \( \ell \). Result \((5.14)\) is shown in a similar way.

**5.3. Discussion.** WENO philosophy desires that our reconstructions be high order accurate when the solution is smooth and yet maintain low order accuracy when there is a discontinuity not in the central cell \( I_i \). For the latter, it is required that \( \epsilon_h = K h^2 \) \cite{1}. We summarize and discuss the results when the solution is either smooth or there is a discontinuity not in the central cell \( I_i \), but the grids are then bounded away from the discontinuity.

Theorems 5.5 and 4.3 together show that, given any \( s \), the standard WENO reconstruction \( R_s \), \( r = 2s - 1 \), behaves as desired. It is high order accurate, i.e., order \( r = 2s - 1 \), when the solution is smooth, and it drops to low order, i.e., order \( s \), when there is a discontinuity not on \( I_i \), provided only that we satisfy a condition on \( \eta \). This condition, given originally in \cite{1}, is that \( \eta \geq s/2 \). We showed that this condition is sharp.

Theorems 5.5 and 4.4 together show that the two-level WENO-AO\((r, s)\) reconstructions can achieve higher order \( r \) accuracy in the smooth case and otherwise
maintain at least order $s$ accuracy. For WENO-JS weights, we simply require that $r \leq 2s - 1$ and $\eta \geq s/2$. WENO-Z weights are more complex, and we require that $\tau$ be chosen so as to have $\tau = \Theta(1)$ in the discontinuous case. Now, we require that $r \leq 2s - 1 + (s - 1) \eta$ and $\eta \geq s/2$, so any $r$ and $s$ can be used, at the expense of requiring $\eta$ to be very large.

For the multilevel WENO-AO$_s(r_\ell, \ldots, r_1, s)$, we have Theorems 4.5 and 5.7. The latter theorem tells us that in the discontinuous case, when the discontinuity lies within the two biggest stencils, the multilevel reconstruction reduces to the base order $s$, independently of how the multiple levels are treated. For WENO-JS weights, we require $r_\ell \leq 2s - 1$ and all the base reconstructions WENO-AO$_s(r_k, s)$ to be accurate, so we are required to take $\eta_{0,k} \geq s/2$. The multilevel reconstruction is no better than the two-level one when WENO-JS weights are used. When WENO-Z weights are used, the accuracy in the smooth case can be as high as we like, provided enough intermediate levels or large enough $\eta$ are taken to satisfy (4.21)–(4.22), and provided $\eta_{0,k}$ are taken so very large that (4.20) holds.

The new multilevel WENO-AO$_s(r_\ell, \ldots, r_1, s)$ behaves better. In the smooth case, we have order $r_\ell$ accuracy provided that the levels satisfy (4.26). Moreover, in the nonsmooth case, the reconstruction drops to a lower order depending on exactly where the discontinuity lies. So if the discontinuity is in stencil $S_0^n$ but not in $S_0^n$ (or some $S_i^n$ when $n = 0$), then we drop to order $r_n$, provided that $\eta_k \geq r_\ell/2$ for each $k$. Since this latter condition forces large values of $\eta$, one can make the reconstruction drop to order $\min(s, r_n)$ for any $s$ provided only that $\eta_k \geq s/2$ for each $k$.

6. A cautionary example of a discontinuity not bounded away from the grid points. Now we consider grids for which the discontinuity $x_\ast$ is not bounded away from the grid points. The following example shows that, in exact arithmetic, the WENO reconstruction may not drop to the accuracy of the smallest stencil, as the philosophy of WENO expects.

Let $u(x) = H(x_\ast - x)$ be a simple step function with a discontinuity at $x_\ast$, where

\[
(6.1) \quad x_\ast = \sum_{k=0}^{\infty} 2^{-2^k} = 0.11010010000000100\cdots
\]

in binary. We use the sequence of grid spacings $\{h_n\}_{n=1}^{\infty}$, where $h = h_n = 2^{-n}$, and grids $\{x_k^h = kh\}_{k=-\infty}^{\infty}$. Let $x_\ast \in [x_m^h, x_{m+1}^h)$ and define $c(h)$ so $x_\ast = x_m^h + c(h)h$, i.e.,

$$
\quad c(h) = \frac{x_\ast - x_m^h}{h} = 2^nx_\ast - m = \sum_{k=0}^{\infty} 2^{n-2^k} - m.
$$

The gridpoints have only $n$ digits after the binary point, so $\liminf_{h \to 0} c(h) = 0$ and the grid points are not bounded away from the discontinuity. Our sequence of grids gives rise to three subsequences as follows.

1. For the subsequence $\{h_n : n = 2^\ell, \ell = 0, \ldots, \infty\}$, we have $c(h_n) = \Theta(h_n)$ as $\ell \to \infty$. That is, $x_\ast$ gets closer to the gridpoints when the grid is refined.

2. For the subsequence $\{h_n : 2^\ell < n < 2^{\ell+1}, \ell = 2, \ldots, \infty\}$, we have $2^{-2^\ell} \leq c(h_n)h_n \leq 2^{-2^\ell+1}$. For each $n$ between its limits $2^\ell$ and $2^{\ell+1}$, we abuse notation by writing $c(h_n) = \Theta(h_n^{-1})$.

3. For the subsequence $\{h_n : n = 2^\ell - 1, \ell = 2, \ldots, \infty\}$, we have $c(h_n) = \Theta(1)$ as $n \to \infty$. This subsequence of grids has $x_\ast$ bounded away from the gridpoints.
We consider the WENO-AO(3,2) reconstruction with WENO-JS weights and \( \epsilon_h = h^2 \). Let
\[
S_0^3 = \{ [x^h_m, x^h_{m+1}], [x^h_{m+1}, x^h_{m+2}], [x^h_{m+2}, x^h_{m+3}] \},
\]
that is, the jump discontinuity lies in the left most cell of \( S_0^3 \). The average on each cell is \( c(h) \), 0, and 0, respectively. Then the stencil polynomials are
\[
P_0^2(x) = -\frac{c(h)}{h} \left( x - x^h_m - \frac{3h}{2} \right), \quad P_0^3(x) = 0,
\]
\[
P_0^3(x) = -\frac{c(h)}{24} - \frac{c(h)}{2h} \left( x - x^h_m - \frac{3h}{2} \right) + \frac{c(h)}{2h^2} \left( x - x^h_m - \frac{3h}{2} \right)^2.
\]
Therefore, the errors at, say, \( x = x^h_m + h = x^h_{m+1} \) are
\[
P_0^2(x^h_{m+1}) - u(x^h_{m+1}) = \Theta(c(h)), \quad P_0^3(x^h_{m+1}) - u(x^h_{m+1}) = 0, \quad P_0^3(x^h_{m+1}) - u(x^h_{m+1}) = \Theta(c(h)), \tag{6.2}
\]
and the smoothness indicators are
\[
\sigma_0^2 = c(h)^2 = \Theta(c(h))^2, \quad \sigma_1^2 = 0, \quad \sigma_0^3 = \frac{4}{3} c(h)^2 = \Theta(c(h))^2. \tag{6.3}
\]
Using (2.7) and (2.8), we compute that
\[
\alpha_0^2 \quad \text{and} \quad \alpha_0^3 = \begin{cases} \Theta(1) & \text{if } c(h) = \Theta(h), \\ \Theta(h^{4\eta}) & \text{if } c(h) = \Theta(h^{-1}), \\ \Theta(h^{2\eta}) & \text{if } c(h) = \Theta(1), \end{cases}
\]
So the error of the reconstruction \( R^{3,2} \) at \( x^h_{m+1} \) is
\[
R^{3,2}(x^h_{m+1}) - u(x^h_{m+1})
\]
\[
= \frac{\alpha_0^3}{\alpha_0^3} \left[ (P_0^3(x^h_{m+1}) - u(x^h_{m+1})) - \sum_{j=0}^{1} \alpha_j^2 (P_j^2(x^h_{m+1}) - u(x^h_{m+1})) \right]
\]
\[
= \Theta(h) \quad \text{if } c(h) = \Theta(h), \quad \Theta(h^{4\eta-1}) \quad \text{if } c(h) = \Theta(h^{-1}), \quad \Theta(h^{2\eta}) \quad \text{if } c(h) = \Theta(1). \tag{6.4}
\]
We illustrate the results (6.2)–(6.4) numerically, using high precision arithmetic. In Figures 6.1–6.3, the black dots are the logarithm of the smoothness indicators and errors to base 2, respectively. The red, green, and blue dashed lines connect the grids in subsequences (1), (2), and (3), where \( c(h) = \Theta(h), \Theta(h^{-1}), \) and \( \Theta(1) \), respectively. The negative value of the slope is the convergence rate, computed over the subsequence.

Figure 6.1 shows that the smoothness indicators of \( \sigma_0^2 \) and \( \sigma_0^3 \) indeed have order 2 in subsequence (1), order \(-2\) in subsequence (2), and constant order in subsequence (3). Figure 6.2 shows the stencil polynomials \( P_0^2 \) and \( P_0^3 \) approximate to order
Fig. 6.1: Log-log plot of smoothness indicators.

Fig. 6.2: Log-log plot of polynomial approximation error.

Fig. 6.3: Log-log plot of the $R^{3,2}$ reconstruction error.
1, −1, and 0 in subsequences (1), (2), and (3), respectively. Figure 6.3 shows the reconstruction \( R^{3,2} \) approximates to only first order in subsequence (1), and this is the best we can guarantee in general. The rate improves in subsequence (2) to order 3 when \( \eta = 1 \) and order 7 when \( \eta = 2 \). As expected for subsequence (3), when the grids are bounded away from the discontinuity, we see order 2 when \( \eta = 1 \), and we see the improved order 4 when \( \eta = 2 \).

We have shown there exist sequences of grids for which the results in Section 5.2 are violated. We are concerned with how often this situation arises. Without loss of generality, consider the same grid spacings \( h = h_n = 2^{-n}, n = 0, 1, \ldots, \infty \). We assume that \( x_\ast \) lies at an arbitrary position within its grid cell, so it is uniformly distributed within this cell in the sense of probability. We need only compute the probability that \( x_\ast \) is like the point in (6.1), i.e., \( c(h) \) is not uniformly bounded below by some positive number. If \( x_\ast \) is written in binary, then we need the number to have an increasing maximum number of consecutive zero digits. We compute the probability

\[
\text{Prob}(x_\ast \text{ has increasing maximum number of consecutive 0's}) = 1 - \text{Prob}(x_\ast \text{ has fixed maximum number of consecutive 0's}) = 1 - \sum_{\ell=0}^\infty \text{Prob}(x_\ast \text{ has maximum number } \ell \text{ of consecutive 0's}) = 1.
\]

This shows that almost surely the gridpoints are not bounded away from the discontinuity, if exact arithmetic is used, as we refine the grid.

This result is not particularly disconcerting for WENO schemes for solving (1.1), however. Trivially, we use finite precision arithmetic, which sets an artificial lower bound on how close \( x_\ast \) can be to the gridpoints. More importantly, however, we solve a given problem on only one or perhaps a few grids, but consider the solution in time. Suppose we arbitrarily set \( c_\ast = 0.001 \). As the shock moves in time, assuming a uniform probability for the shock position with respect to the grid, there is a 99.8% chance that the discontinuity \( x_\ast \) is bounded away from the gridpoints (independently of \( h \)). That is, within a WENO scheme, the situation described in this cautionary example does not arise often. It is already clear that there is a single big stencil with center cell \( I_i \) containing \( x_\ast \) for which \( u \) is not smooth on any small stencil, so WENO reconstruction does not give a good result on that cell. The example shows that there may be a few other cells arising from time to time that have poor approximation.

WENO reconstruction still captures the shock, as is well-known from numerical tests. In fact, we saw above that as \( h \to 0 \), if \( \sigma \to 0 \) on some stencil \( S^r \), then \( P^r \) converges smoothly to a constant. WENO reconstruction will include this stencil, but picking it up will give a good reconstruction, albeit not to the order we had desired.

7. Numerical results in one space dimension. In our one dimensional tests of the conservation law (1.1), the \( L^1 \) and \( L^\infty \) errors are computed, respectively, by

\[
\sum_i \left| \frac{1}{\Delta x_i} \int_{I_i} u(x, t^n) \, dx - \bar{u}_i^n \right| \Delta x_i \quad \text{and} \quad \max_i \left| \frac{1}{\Delta x_i} \int_{I_i} u(x, t^n) \, dx - \bar{u}_i^n \right|.
\]

7.1. Reconstruction near jump discontinuities. Our first test case is from [6, 1, 8]. Recall that \( H \) is the Heaviside function. For \( x_\ast \) fixed, let

\[
u(x) = g(x) + H(x_\ast - x).
\]

Consider the grid spacings \( \{h_n\}_{n=0}^\infty \), where \( h_n = 0.1/2^n \), and fix \( I_i = [0, h_n] \). We test the accuracy of and WENO-AO(9, 7, 5, 3) and WENO-AO(9, 7, 5, 3) reconstructions
at $x = 0$ when $x_\ast = -4h, -3h, -2h$, and $-h$. That is, $u$ is smooth only on $S^9_0, S^7_0, S^5_0$, and $S^3_j$, respectively, where $j = 0, 1$.

We first take $g(x) = x^3 + \sin(x)$, so $g'(0) \neq 0$. The results are shown in Table 7.1 for WENO-JS weights and Table 7.2 for WENO-Z weights. We set our algorithm parameters based on Theorems 5.5 and 5.7 (see also the discussion in Section 5.3). We take $\epsilon_h = h^2$ for both of WENO-JS and WENO-Z weights.

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Table 7.1: Example 7.1 with $g(x) = x^3 + \sin(x)$. WENO-JS weights with $\epsilon_h = h^2$. Error and convergence rate at $x = 0$.

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Table 7.2: Example 7.1 with $g(x) = x^3 + \sin(x)$. WENO-Z weights with $\epsilon_h = h^2$. Error and convergence rate at $x = 0$.

For WENO-AO$_3(9, 7, 5, 3)$, we use in the two-level base reconstructions the linear weights 0.85 for the big stencil and 0.05 for the three small stencils, and $\eta_{0,k} = 2 > s/2 = 3/2$. For the recursive levels, the higher order linear weight is 0.85 and 0.15 is the other weight. For the recursive level $r_2 = 7$, $\eta_2 = 1 = (r_1 - s)/2$, where $r_1 = 5$. For level $r_3 = 9$, $\eta_3 = 2 = (r_2 - s)/2$.

WENO-AO$_3(9, 7, 5, 3)$ drops to the accuracy of the base level 3 when $u$ is not smooth on $S^7_0$, i.e., for the latter two values of $x_\ast$. Since $r_{\text{max}} = 2s - 1 = 5$ in Theorem 4.4, we conclude that WENO-AO$_3(9, 7, 5, 3)$ is only $O(h^2)$ for WENO-JS weights when $u$ is smooth on $S^9_0$ or the biggest stencil $S^9_0$. For WENO-Z weights, hypotheses (4.20)–(4.22) are satisfied, so the reconstruction achieves optimal order if
Accuracy of WENO and Adaptive Order WENO Reconstructions

u is smooth on S_7^0 or S_9^0. We see exactly these results in Tables 7.1–7.2.

For the new WENO-AO(9, 7, 5, 3), we take α_j^3 = 0.05, α_r^k = 0.85, and η_k = ⌈r_k^2⌉, k = 1, 2, 3. Since (4.26) holds, the new reconstruction is the optimal order r_k, k ≥ 1, for each recursive level if u is smooth on S_r^k, for both WENO-JS and WENO-Z weights. Again, we see these results in Tables 7.1–7.2.

For direct comparison to [6, 1, 8], we also show the results for g(x) = x^3 + cos(x).

Note that g′(0) = 0, so we are at a critical point and (2.5) shows that we may have somewhat better results. Indeed, we see some improvement in the order of accuracy, but the general trend in the results agrees with the previous case when there is no critical point.

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Table 7.3: Example 7.1 with g(x) = x^3 + cos(x). WENO-JS weights with ϵ_h = h^2.

Error and convergence rate at x = 0.

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Table 7.4: Example 7.1 with g(x) = x^3 + cos(x). WENO-Z weights with ϵ_h = h^2.

Error and convergence rate at x = 0.

7.2. Burgers’ equation. We next solve Burgers’ equation u_t + (u^2/2)_x = 0 with the initial condition u_0(x) = 0.25 + 0.5 sin(πx) on [-1, 1] to the time T = 0.25. A shock forms in the solution after this time, but the solution sharpens up to time T so as to have a very steep front. The exact solution can be determined, and the convergence results are shown in Tables 7.5 and Tables 7.6. We use the same parameters as in Example 7.1.
Table 7.5: Burger’s equation. WENO-JS weights with $\epsilon_h = h^2$. Error and convergence rate on uniform grids at time $T = 0.25$.

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<td></td>
<td>WENO-AO(9, 5, 3)</td>
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<td>40</td>
<td>2.50E-06</td>
<td>5.34</td>
<td>1.24E-05</td>
<td>4.69</td>
<td>1.92E-06</td>
<td>5.59</td>
<td>9.34E-06</td>
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<td>4.36E-07</td>
<td>4.83</td>
<td>1.82E-09</td>
<td>8.04</td>
<td>1.04E-08</td>
<td>8.06</td>
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<tr>
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<td>3.11E-09</td>
<td>4.92</td>
<td>1.43E-08</td>
<td>4.93</td>
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<td>9.85E-11</td>
<td>4.98</td>
<td>4.51E-10</td>
<td>4.99</td>
<td>5.75E-14</td>
<td>8.75</td>
<td>9.97E-14</td>
<td>8.73</td>
</tr>
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</table>

Table 7.6: Burger’s equation. WENO-Z weights with $\epsilon_h = h^2$. Error and convergence rate on uniform grids at time $T = 0.25$.

<table>
<thead>
<tr>
<th>N</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
<th>$L^1$ error</th>
<th>order</th>
<th>$L^\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td>WENO-AO(9, 7, 5, 3)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>40</td>
<td>2.50E-06</td>
<td>5.32</td>
<td>1.23E-05</td>
<td>4.65</td>
<td>1.92E-06</td>
<td>6.69</td>
<td>3.07E-06</td>
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<tr>
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<tr>
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<td>1.41E-14</td>
<td>8.75</td>
<td>9.97E-14</td>
<td>8.73</td>
</tr>
</tbody>
</table>

Because Theorem 4.5 caps the order of accuracy at $2s - 1 = 5$ when using WENO-JS weights, we see that the old WENO-AO$_3$ reconstruction is only fifth order accurate. It is nearly optimal using WENO-Z weights. The new WENO-AO reconstruction performs similarly using WENO-Z weights, but improves the solution with WENO-JS weights. With these weights, we see nearly optimal results for WENO-AO(7, 5, 3) and WENO-AO(9, 7, 5, 3), but WENO-AO(9, 5, 3) seems to be only perhaps seventh order accurate. This is predicted by the condition (4.25) of Theorem 4.6, i.e., the maximum rate is capped at $5 + 3 - 1 = 7$. 
7.3. The one dimensional Euler system. The one dimensional Euler system of gas dynamics is given by

\[
\frac{\partial}{\partial t}\left(\begin{array}{c}
\rho \\
m \\
E\\
\end{array}\right) + \frac{\partial}{\partial x}\left(\begin{array}{c}
mu^2 + p \\
u(E + p)\\
\end{array}\right) = 0,
\]

where \( m = \rho u \), \( E = p/((\gamma - 1)) + \rho u^2/2 \) and \( \rho, u, m, p, \) and \( E \) are the density, velocity, momentum, pressure, and energy, respectively, and \( \gamma = 1.4 \). We compare the two WENO-AO reconstructions on two of the more challenging standard test problems.

Following [2], let \( \gamma_{Hi} = \gamma_{Lo} = 0.85 \). For WENO-AO\(_{(7,5,3)}\), take

\[
\alpha^5_0 = \gamma_{Hi}, \quad \alpha^3_{-1} = \alpha^3_1 = (1 - \gamma_{Hi})(1 - \gamma_{Lo})/2, \quad \alpha^3_0 = (1 - \gamma_{Hi})\gamma_{Lo};
\]

\[
\gamma^{7,3} = \gamma_{Hi}, \quad \gamma^{5,3} = 1 - \gamma_{Hi};
\]

\[
\eta_{0,1} = \eta_{0,2} = 2, \quad \eta_2 = 1.
\]

For WENO-AO\(_{(7,5,3)}\), take

\[
\alpha^7_0 = \alpha^5_0 = \gamma_{Hi}, \quad \alpha^3_{-1} = \alpha^3_1 = (1 - \gamma_{Hi})(1 - \gamma_{Lo})/2, \quad \alpha^3_0 = (1 - \gamma_{Hi})\gamma_{Lo};
\]

\[
\eta_1 = 2, \quad \eta_2 = 3.
\]

For both reconstructions, we use \( \epsilon = h^2 \).

7.3.1. Shu and Osher’s shock interaction with entropy waves. The shock interaction with entropy waves problem given in [12] has a moving Mach 3 shock interacting with sine waves in the density. The initial condition is

\[
(\rho, u, p) = \begin{cases} 
\rho_l = 3.587143, \ u_l = 2.629396, \ p_l = 10.333333, & \text{for } 0 < x < 0.1, \\
\rho_r = 1 + 0.2\sin(5(10x - 5)), \ u_r = 0, \ p_r = 1, & \text{for } 0.1 < x < 1,
\end{cases}
\]

We compute the density at \( T = 0.16 \) using \( \Delta t = 0.1\Delta x \) and \( N = 400 \) cells. The results are shown in Figure 7.1. We see little difference between the two reconstructions, although perhaps the new one reaches the peaks of the sine waves slightly better.

Fig. 7.1: Shu and Osher’s shock interaction. The density at \( T = 0.16 \) using \( N = 400 \) cells. The plots are the reference solution (green line), WENO-AO\(_{(7,5,3)}\) (blue squares) and WENO-AO\(_{(7,5,3)}\) (red circles) with WENO-Z weights.
7.3.2. Woodward and Colellas double blast test. The last test uses the initial condition

\[
(\rho, m, E) = \begin{cases} 
\rho_l = 1, & m_l = 0, \ E_l = 1000/(\gamma - 1), \text{ for } 0 < x < 0.1, \\
\rho_m = 1, & m_m = 0, \ E_m = 0.01/(\gamma - 1), \text{ for } 0.1 < x < 0.9, \\
\rho_r = 1, & m_r = 0, \ E_r = 100/(\gamma - 1), \text{ for } 0.9 < x < 1.
\end{cases}
\]

Two shock waves form and interact before time \( T = 0.038 \), so this is a particularly challenging example. The density at time \( T = 0.038 \) is shown in Figure 7.2. The new reconstruction captures the solution a bit better.

REFERENCES