On non-existence of Lebesgue-like measures in infinite dimension

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Abstract
We briefly comment on the issue of the non-existence of an analogue of the Lebesgue measure in infinite-dimensional Banach spaces.

1 Notation
Let \((X, \|\cdot\|)\) be a normed linear space (with \(\mathbb{R}\) or \(\mathbb{C}\) as the base field). In what follows \(B(x, r)\) denotes the open ball centered at \(x\) with radius \(r\). We denote by \(S_1\) the unit sphere centered at the origin, \(S_1 = \{x \in X : \|x\| = 1\}\). Moreover, for a subspace \(M \subseteq X\) and a point \(p \in X\), we define
\[
\text{dist}(p, M) = \inf_{m \in M} \|p - m\|.
\]

2 Riesz’s Lemma

Lemma 2.1 (Riesz). Let \((X, \|\cdot\|)\) be a normed linear space and suppose \(M\) is a proper closed subspace of \(X\). Then for any \(\nu \in (0, 1)\) there exists \(x_\nu \in S_1\) such that \(\|x_\nu - m\| \geq \nu\) for all \(m \in M\).

Proof. Let \(p \in X \setminus M\) and note that \(d := \text{dist}(p, M) > 0\). Take \(m_0 \in M\) such that \(\|p - m_0\| \leq d/\nu\). Now, set \(x_\nu = \frac{p - m_0}{\|p - m_0\|}\). Thus, \(\|x_\nu\| = 1\) and for every \(m \in M\), we have,
\[
\|x_\nu - m\| = \left\| \frac{p - m_0}{\|p - m_0\|} - m \right\| = \frac{1}{\|p - m_0\|} \left\| p - \left( m_0 + \frac{p - m_0}{\|p - m_0\|} m \right) \right\| \geq d/(d/\nu) = \nu.
\]

The following technical result is a consequence of Riesz’s Lemma. The result is stated for \(B(0, 1)\) but can be easily generalized for any open ball. The following result shows why we cannot define an analogue of the Lebesgue measure in an infinite-dimensional separable Banach space.

Lemma 2.2. Let \(X\) be an infinite dimensional normed linear space. Then there exists a countably infinite collection of disjoint balls \(B(x_n, \varepsilon)\) inside \(B(0, 1)\).

Proof. Let \(y_1 \in S_1\) and let \(M_1 = \text{span}\{y_1\}\). Now by Riesz’s Lemma, we know there exists \(y_2 \in S_1\) such that \(\|y_2 - m\| \geq 1/2\) for all \(m \in M_1\). We let \(M_2 = \text{span}\{y_1, y_2\}\) and proceeding inductively, get \(y_3, y_4, y_5, \ldots\), such that \(y_n \in S_1\) for all \(n\) and for subspaces
\[
M_n = \text{span}\{y_1, \ldots, y_n\}.
\]

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Last revised: February 18, 2014
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we have dist\( (y_{n+1}, M_n) \geq 1/2 \). The successive application of Riesz’s Lemma is justified because for all \( n, M_n \) is finite-dimensional and is hence a proper closed subspace of \( X \). Note for the sequence \( \{y_n\}_{n=1}^\infty \) we have, \( y_n \in S_1 \) and \( \|y_{n+1} - y_n\| \geq 1/2 \) for all \( n \in \mathbb{N} \); the latter also implies, \( B(y_n, 1/4) \cap B(y_{n+1}, 1/4) = \emptyset \). Hence, the statement of the lemma holds with the collection of balls given by \( \{B(x_n, \varepsilon)\}_{n=1}^\infty \), where \( x_n = \frac{1}{2} y_n \) and \( \varepsilon = 1/8 \).

### 3 Measures on Banach spaces

For any Borel measure \( \mu \) on \( X \) to behave like the Lebesgue measure it must be a translation invariant positive measure which assigns a finite measure to open balls.

**Proposition 3.1.** Let \( X \) be an infinite dimensional separable Banach space. Then there exists no non-trivial translation invariant positive Borel measure \( \mu \) on \( X \) which is finite on open balls.

**Proof.** Suppose \( \mu \) is a translation invariant positive Borel measure which assigns finite measures to open balls. By Lemma 2.2 we know that \( B(0, 1) \) contains a countably infinite collection of disjoint balls \( \{B(x_n, \varepsilon)\}_{n=1}^\infty \). Then, by translation invariance, \( \mu(B(x_n, \varepsilon)) \) is the same for every \( n \in \mathbb{N} \), \( \mu(B(x_n, \varepsilon)) = \alpha \) with \( \alpha \in [0, \infty) \). Now if \( \alpha > 0 \) then we have \( \mu(B(0, 1)) \geq \mu(\cup_n B(x_n, \varepsilon)) = \sum_1^\infty \mu(B(x_n, \varepsilon)) = \sum_1^\infty \alpha = \infty \), which is a contradiction. Note that we also used the fact that \( \{B(x_n, \varepsilon)\}_{1}^\infty \) are disjoint. On the other hand, if \( \alpha = 0 \), then by separability we can cover the whole space \( X \) with open balls of radius \( \varepsilon \) and get that \( \mu(X) = 0 \); i.e., \( \mu \) is the trivial (zero) measure.

### 4 Remark

The argument leading to the result on non-existence of an analogue to the Lebesgue measure in infinite-dimension is related to the argument showing that the Heine-Borel Theorem does not hold in infinite-dimensional normed linear spaces. In particular, in the argument in the proof of Lemma 2.2 we use Riesz’s Lemma to construct a sequence \( \{y_n\}_{n=1}^\infty \in B(0, 1) \) which satisfies \( \|y_n - y_m\| \geq 1/2 \) for \( n \neq m \); clearly this sequence cannot have any convergent subsequence and thus \( B(0, 1) \) is not compact. It is interesting that while the result concerning the noncompactness of the closed unit ball in infinite dimensional normed linear spaces is usually encountered in a first course in functional analysis, the former result, regarding the non-existence of an equivalent of a Lebesgue measure in infinite dimension is usually mentioned in much more advanced treatments of probability.