CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2023, Final Exam, 5:00-8:00 p.m., Fri, Dec 8, POB 6.304

1. A linear algebra "sanity check".

   Consider $\mathbb{R}^3$. Let A be the mirror (symmetry) transformation with respect to plane $x_1 - x_3 = 0$ (think about a mirror placed in this position and the transformation that maps a point P into its mirror image P').

   (a) Is A a linear map ? Explain (2 points).

   (b) Write down the matrix representation for map A in the canonical basis (3 points).

   (c) Explain why all linear maps from $\mathbb{R}^3$ into itself - $L(\mathbb{R}^3,\mathbb{R}^3)$, form a vector space. What is the dimension of the space (2 points) ?

   (d) Do the mirror transformations (with respect an arbitrary plane passing through the origin) form a vector subspace of $L(\mathbb{R}^3,\mathbb{R}^3)$? Explain your answer. If yes, what is the dimension of this subspace (3 points) ?

   (e) Define adjoint for a linear operator in a general Hilbert setting (2 points).

   (f) Compute the adjoint of map A with respect to the canonical inner product in $\mathbb{R}^3$. Is A self-adjoint ? (3 points).

   (g) Define an orthonormal matrix (2 points).

   (h) Is the matrix representation of map A (any mirror transformation) in the canonical basis an orthonormal matrix ? Explain, why ? (3 points)

   Answers:

   (a) Yes, it is. Operations of taking a mirror image and multiplication by a number, commute. Similarly, vector addition and the mirror image map commute as well.

   (b) This is really a 2D problem. By inspection, the mirror map is:

   $$(x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1)$$

   or,

   $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c) Functions defined on any set (in our case $\mathbb{R}^3$) with values in a vector space (in our case $\mathbb{R}^3$) equipped with pointwise addition and scalar multiplication, form a vector space. One has only to argue that the linear maps form a subset closed with respect to the vector space operations and, therefore, form a vector subspace of all functions defined on $\mathbb{R}^3$. This follows from the fact that a linear combination of linear maps is a linear map itself. Dimension of $L(X, Y)$ is always equal to the product of dim $X = n$ and dim $Y = m$ (in our case = 9). This follows from the isomorphism between $L(X, Y)$ and $m \times n$ matrices.

(d) Well, they do not. One possible way to see this, is to develop the matrix representation for a general mirror map corresponding to an arbitrary plane,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0, \quad a_1^2 + a_2^2 + a_3^2 = 1\,.$$

Equation for a straight line passing through point (x_1, x_2, x_3) and orthogonal to the plane, is:

$$\begin{aligned} y_1 &= x_1 + \lambda a_1 \\ y_2 &= x_2 + \lambda a_2 \\ y_3 &= x_3 + \lambda a_3 \end{aligned}$$

The value of parameter λ corresponding to the intersection point with the plane is obtained by plugging the formulas for y_i into the equation of the plane, we get:

$$\lambda = -(a_1x_1 + a_2x_2 + a_3x_3)\,.$$

The mirror image of point x is obtained now by doubling the value of the parameter,

$$\begin{aligned} y_1 &= x_1 - 2(a_1x_1 + a_2x_2 + a_3x_3)a_1 \\ y_2 &= x_2 - 2(a_1x_1 + a_2x_2 + a_3x_3)a_2 \\ y_3 &= x_3 - 2(a_1x_1 + a_2x_2 + a_3x_3)a_3 \end{aligned}$$

The corresponding matrix representation is:

$$\begin{pmatrix} 1-2a_1^2 & -2a_1a_2 & -2a_1a_3 \\ -2a_1a_2 & 1-2a_2^2 & -2a_2a_3 \\ -2a_1a_3 & -2a_2a_3 & 1-2a_3^2 \end{pmatrix}$$

Due to the fact that coefficients a_i come from a unit sphere in $\mathbb{R}^3$, and the nonlinear dependence of the matrix wrt to the coefficients, it is easy to see that maps of this type do not form a vector space. More formally, we can argue that such matrices do not form a set closed with respect to multiplication by a scalar. Indeed, take the first column for our case, i.e. for $a_1 = a_2 = 1/\sqrt{2}, a_3 = 0$, and multiply it by a factor of two to obtain $(0, -2, 0)^T$. Can we find a plane, i.e. coefficients a_i such that the general formula will yield these values, i.e.,

$$1 - 2a_1^2 = 0 \quad -2a_1a_2 = 2 \quad -2a_1a_3 = 0\,?$$

We get $a_1 = \pm 1/\sqrt{2}, a_3 = 0$ and then $a_2 = \mp \sqrt{2}$. But a_i must represent components of a unit vector, so the value for a_2 is not acceptable.

There are many other ways to convince yourself that the mirror images do not form a vector space.

(e) The notion of the adjoint involves two Hilbert spaces X and Y with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$. Given a linear map $A : X \to Y$, we define the adjoint map $A^* : Y \to X$ by:

$$A^* = R_X^{-1} A^T R_Y$$

where $A : Y^* \to X^*$ is the transpose of A, and R_X, R_Y are Riesz maps for X and Y, resp. Equivalently,

$$(Ax, y)_Y = (x, A^* y)_X \quad x \in X,\, y \in Y\,.$$

(f) Nothing to compute. The matrix is symmetric so the operator is self-adjoint, i.e. $A^* = A$.

(g) Matrix A is *orthonormal* if $A^{-1} = A^T$.

(h) Yes, it is.

2. An integration exercise.

   (a) State the Monotone Convergence Theorem for non-negative functions and the Lebesgue Dominated Convergence Theorem for non-negative functions (6 points).

   (b) Let f be a real-valued, positive function, defined on interval $[a, \infty), a > 0$. Recall the definition of the classical singular integral,

   $$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx\,.$$

   Prove that if f is Lebesgue integrable then the singular integral exists, and it coincides with the Lebesgue integral (over $[a, \infty)$) (6 points).

   (c) Is the converse true ? In other words, if f is Lebesgue integrable over any interval $[a, b)$, and the limit exists, can we conclude that f is Lebesgue integrable over the infinite integral $[a, \infty)$? Prove or disprove (6 points).

   (d) Provide an example of a concrete function $f(x)$ that illustrates the discussion (2 points).

Answers:

(a) See the book.

(b) We can restrict ourselves to integer $b = n = 1, 2, \ldots$. Define,

$$f_n(x) := \begin{cases} f(x) & x \le n \\ 0 & x > n \end{cases}$$

Then $f_n \to f$ pointwise, and the sequence is dominated by f. Consequently, by the Lebesgue Theorem,

$$\int_a^n f = \int_a^\infty f_n \to \int_a^\infty f\,.$$

(c) Yes. By the Monotone Convergence Lemma,

$$\int_a^\infty f = \int_a^\infty \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^\infty f_n = \lim_{n\to\infty} \int_a^n f\,.$$

(d) $f(x) = 1/x^2, a = 1$ would do.

3. A topology problem.

   (a) Define a compact set, and a sequentially compact set, both in an arbitrary Hausdorff topological space. Under what assumptions on the topological space, the two notions are equivalent ? (5 points)

   (b) Formulate the four fundamental properties for both compact andf sequentially compact sets. Prove any two of the properties for sequentially compact sets. (5 points).

   (c) Consider the following minimization problem,

   $$\min_{x \in K} f(x)$$

   where

   $$K := \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 0,\, x_2 \geq 0,\, x_1 + x_2 \leq 1\}$$

   and

   $$f(x) = \ln^2(x_1^2 + x_2^2)\,.$$

   Is this problem well posed, i.e. , does a minimum exist ? (10 points)

Answers:

(a) See the book

(b) See the book

(c) Set K is closed and bounded, so it is compact. But you cannot use directly the Weierstrass Thm argument as the function f is *not* real-valued on the whole set, at 0 it is ∞. But we are interested in the infimum only. We can cut out of K a small ball $B(0, \epsilon)$. As $f(x) > \ln^2 \epsilon^2$ in the ball, and f does take smaller values in the rest of K, we can replace original K with the modified K and then shoot the Weierstrass argument.

   Alternatively, you can simply notice that the minimum is attained at the two corner points and it is equal zero.

4. Metric spaces.

    - Provide definition of a metric, and a metric space (3 points).
    - Let (X, d) be an arbitrary metric space. Prove that a new function,

$$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

      is also a metric on X (10 points).
    - Are the two metrics equivalent (for a general metric d) ? Are they topologically equivalent ? Explain, why (7 points) ?

Answers:

- See the book.
- See the book.
- The metrics cannot be equivalent in general. Metric ρ is always bounded whereas metric d need not be. But they are topologically equivalent as the corresponding bases of neighborhoods are equivalent.

5. Contraction Maps. Consider the following Initial-Value Problem (IVP):

$$\begin{cases} \dfrac{dq}{dt} = t \ln q(t), \quad t > 0 \\ q(0) = 1 \end{cases}$$

- State Banach Contractive Map Theorem (3 points).

  Answer:
  Let (X, d) be a complete metric space. Let $D \subset X$ (then (D, d) is itself a metric space, too...), and $A : D \to D$ is a contraction, i.e.

  $$d(A(f), A(g)) \leq k\, d(f, g), \quad \forall f, g \in D, \quad k < 1$$

  Then function A has a unique fixed point in set D.

- Use the theorem to prove local existence and uniqueness of solution to the IVP, i.e. that there exists an interval $(0, T)$ in which the equation is satisfied. Provide a *concrete* value of T (17 points).

  Solution:
  The problem is equivalent to the solution of the integral equation:

  $$q(t) = 1 + \int_0^t s \ln(q(s))\, ds$$

  Consider the Chebyshev space $C[0, T]$ (with unknown T at this point...) and define the map A using the right-hand side of the equation above:

  $$(Aq)(t) = 1 + \int_0^t s \ln(q(s))\, ds$$

  First of all, we need to define a set $D \subset C[0, T]$ such that map A sets the set D into itself. Assume that $q(t)$ will vary in the box:

  $$D = \{q \in C[0, T] \,:\, e^{-1} \leq q(t) \leq e, \quad 0 \leq t \leq T\} \tag{0.1}$$

  (notice that the box includes the initial value $q = 1$). Then $-1 \leq \ln q(t) \leq 1$, i.e. $|\ln q(t)| \leq 1$. Consequently,

  $$\left| \int_0^t s \ln q(s)\, ds \right| \leq \int_0^t s\, ds = \frac{1}{2}t^2$$

  so,

  $$|(Aq)(t) - 1| \leq \frac{1}{2}T^2$$

This gives two bounds for T. From the right:

$$(Aq)(t) \leq 1 + \frac{1}{2}T^2 \leq e \quad \Rightarrow \quad T \leq \sqrt{2(e-1)}\,,$$

and from the left:

$$e^{-1} \leq 1 - \frac{1}{2}T^2 \leq (Aq)(t) \quad \Rightarrow \quad T \leq \sqrt{2(1-e^{-1})}\,.$$

Now, map A must be a contraction. With flux $F(s,q) = s\ln q$,

$$|\frac{\partial F}{\partial q}| = s|\frac{1}{q}| \leq es$$

so, with q coming form box (**??**), the flux satisfies the Lipschitz condition:

$$|F(s,q_1) - F(s,q_2)| \leq es|q_s - q_2|\,.$$

This leads to the estimate;

$$|(Aq_1)(t) - (Aq_2)(t)| \leq \int_0^t es\,ds\,\|q_1 - q_2\|_{C[0,T]} \leq \frac{e}{2}T^2\,\|q_1 - q_2\|_{C[0,T]}\,.$$

Consequently, a sufficient condition for a contraction is

$$T < \sqrt{\frac{2}{e}}\,.$$

In conlusion, the IVP will have a unique solution for

$$T < \min\{\sqrt{2(e-1)}, \sqrt{2(1-e^{-1})}, \sqrt{\frac{2}{e}}\}\,.$$